



**Yale ICF Working Paper No. 02-08**

November 2004

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First Draft: September 2000  
Latest Draft: November 2004

This paper has benefited from the comments an anonymous referee and numerous colleagues including the participants at the Five Star Conference hosted by New York University, the Hedge Fund Conference hosted by Borsa Italiana, the Berkeley Program in Finance, and Louisiana State University.

# Sharpening Sharpe Ratios

It is now well known that the Sharpe ratio and other related reward-to-risk measures may be manipulated with option-like strategies. This paper derives the general conditions for achieving the maximum expected Sharpe ratio. Also derived are static rules for achieving the maximum Sharpe ratio with two or more options, as well as a continuum of derivative contracts. The optimal strategy has a truncated right tail and a fat left tail. Additionally, the paper provides dynamic rules for increasing the Sharpe ratio.

In order to address the sensitivity of the Sharpe ratio to information-less, option-like strategies, the paper proposes an alternative measure that is less susceptible to such manipulations. The case for using this alternative ranking metric is particularly compelling in the hedge fund industry where the use of derivatives is unconstrained and manager compensation itself induces a non-linear payoff.

# Sharpening Sharpe Ratios

## 1. Introduction

The Sharpe ratio is one of the most common measures of portfolio performance. William Sharpe developed it in 1966 as a tool for evaluating and predicting the performance of mutual fund managers. Since then the Sharpe ratio, and its close analogues the Information ratio, the squared Sharpe ratio and M-squared, have become widely used in practice to rank investment managers and to evaluate the attractiveness of investment strategies in general. The appeal of the Sharpe measure is clear. It is an affine transformation of a simple t-test for equality in means of two variables, the first variable being the manager's time series of returns and the second being a benchmark.<sup>1</sup> The Sharpe ratio is also ubiquitous in academic research as a metric for bounding asset prices.<sup>2</sup>

Unfortunately, the Sharpe ratio is prone to manipulation — particularly by strategies that can change the shape of probability distribution of returns. For example, Henriksson and Merton (1981) and Dybvig and Ingersoll (1982) show that non-linear payoffs limit the applicability of the Sharpe ratio to the problem of performance evaluation. More recently, Bernardo and Ledoit (2000) have shown that Sharpe ratios are particularly misleading when the shape of the return distribution is far from normal.<sup>3</sup> Lo (2002) and Getmansky, Lo, and Makarov (2003) show how smoothing returns over time can inflate the Sharpe ratio. Spurgin (2001) shows that managers can improve their expected Sharpe ratio by selling off the upper end of the potential return distribution. A similar result can be found in Ferson and Siegel (2001). They analyze a manager's optimal response when he has some forecasting abilities and is judged via a mean variance measure such as the Sharpe ratio. Their analysis shows that the optimal portfolio cuts its holdings of risky assets when they have high expected returns in order to help reduce the realized portfolio variance. Other researchers, recognizing the limitations of the Sharpe ratio and its relatives, have sought alternatives to the reward-to-variability approach. These include stochastic-discount factor based performance measures (c.f. Chen and Knez (1996)) and more direct measures of active management skill (c.f. Grinblatt and Titman (1992)). The literature on performance evaluation is a large one (c.f. Brown, 2000), and much of it has focused on the limitations of standard measures. However, despite twenty years of academic understanding of the problems of benchmarking and performance measurement, the Sharpe ratio and its relatives remain fundamental tools in research and practice.

In this paper we identify a class of strategies that maximize these performance measures, without requiring any forecasting skill. We derive rules for achieving the

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<sup>1</sup> For a review of its history and use, see Sharpe (1994). For a current textbook discussion and applications of the Sharpe ratio, see for example, Bodie, Kane and Marcus (1999) p. 754-758, and back endsheet. For applications in the mutual fund industry, see Morningstar (1993) p. 24.

<sup>2</sup> See Cochrane and Saa-Requeno (1999) for a discussion of the application of Sharpe ratios to current asset pricing research.

<sup>3</sup> To address this problem they propose a semi-parametric alternative biased on the gain-loss ratio that, in effect discards the information in the tails of the distribution.

maximum Sharpe ratio when the manager has the freedom to take positions in derivative securities, and when the manager has a given history of returns. Our analysis shows that the best static manipulated strategy has a truncated right tail and a fat left tail. The optimal strategy involves selling out-of-the-money calls and selling out-of-the-money puts in an uneven ratio that insures a regular return from writing options and a large exposure to extreme negative events. We also show that the best dynamic strategy for maximizing the Sharpe ratio involves leverage conditional upon underperformance.

We next address the problem of gaming the Sharpe ratio using option-like or dynamic strategies by identifying the necessary conditions under which a manipulation-free measure exists. The first hurdle is to define what is meant by a manipulation-free measure. This paper proposes three conditions that should be met and protect against the types of strategies explored here: (1) an uninformed agent cannot increase the measure's value by deviating from the benchmark portfolio for any past performance history, (2) the measure does not depend on the initial value of the portfolio, and (3) if returns are log normal and the market portfolio is efficient, then an uninformed agent can maximize the measure's value by holding the market portfolio.

From condition (1) the measure must be a utility function (or transformation thereof) and it must be additive over time. From (2) it must be a power utility function, and (3) then pins down the risk aversion coefficient. Thus, the paper shows that if a manipulation proof measure exists it is in the form of the realized value of a power utility function with the risk aversion coefficient set to make the market portfolio optimal. If these conditions are not met, we show that there is no non-manipulable portfolio ranking measure.

The time-separable power utility function gives us a new performance ranking measure that is thus immune to informationless manipulation. We can use this measure to test the historical performance of the Sharpe ratio. Results of our empirical tests suggest that the Sharpe ratio does a good job, on average, at ranking mutual fund and hedge fund portfolios. This in fact may explain why the Sharpe ratio continues to be used in practice despite known limitations. The rank correlations between the utility-based measure and the Sharpe ratio are high; however they can differ significantly depending upon the skewness of the return distribution, and upon the time-period of study. Evidence from the hedge fund universe is consistent with some funds using option-like strategies to enhance their Sharpe ratio. In these circumstances, ranking managers according to a utility-based measure is superior.

The results have a number of implications for investment management. Interest in alternative investments has grown dramatically in the past decade. Hedge funds in particular have attracted interest by institutional managers and high net worth individuals. These investment vehicles have broad latitude to invest in a range of instruments including derivative securities. Mitchell and Pulvino (2001) document that merger arbitrage, a common hedge fund strategy, generates returns that resemble a short put-short call payoff. Recent research by Agarawal and Naik (2001) shows that hedge fund managers in general follow a number of different styles that are nonlinear in the returns to relevant indices. In a manner similar to Henriksson and Merton, Agarawal and Naik use option-like payoffs as regressors to capture these non-linearities. In fact, option-like payoffs are inherent in the compensation-structure of the typical hedge fund contract. Goetzmann, Ingersoll and Ross (2001) show that

the high water mark contract, the most common compensation contract in the hedge fund industry, effectively leaves the investor short 20% of a call option. The call is at-the-money each time it is “reset” by a payment and out-of-the money otherwise.

The paper is structured as follows. Section 2 derives the maximal Sharpe ratio in a static setting under a range of conditions from complete markets to strategies constrained to holding a pair of standard puts and calls. Section 3 deals with the additional problems caused by permitting dynamic manipulation of the portfolio to improve the Sharpe ratio. Section 4 develops a manipulation-free measure of performance and compares it empirically with the Sharpe ratio on mutual fund and hedge fund returns. Section 5 discusses the further implications of the results, and Section 6 concludes.

## 2. Using Financial Derivatives to Maximize the Sharpe Ratio

The growth and increasing complexity of financial derivatives available has made the alteration of portfolio returns increasingly simple to accomplish. This is straightforward if a fund's primary purpose is index enhancement, as index options are readily available. This section of the paper analyzes Sharpe manipulation using financial derivatives.

### 2A. The Maximal Sharpe Ratio in a Complete Market

First consider a market that is complete over all price outcomes or can be made so with dynamic trading. The standard single-period portfolio problem would be to maximize expected utility,  $\sum p_i u(w_i)$ , where  $p_i$  is the probability for state  $i$ ,  $u(\cdot)$  is the utility function of the investor, and  $w_i$  is the total return (or wealth) in state  $i$ . The optimal portfolio,  $w_i^o$ , is characterized by

$$u'(w_i^o) = \theta \hat{p}_i / p_i \tag{1}$$

where  $\theta = \mathbb{E}[u'(w^o)]$  is the Lagrange multiplier from the budget constraint, and  $\hat{p}_i$  is the risk-neutral probability of state  $i$ .<sup>4</sup>

Suppose instead that an investor wishes to form a portfolio with the largest possible Sharpe ratio. Any portfolio can be decomposed into a risk-free asset plus a risky zero-investment (or arbitrage) portfolio,  $\tilde{x}$ , of excess returns. The Sharpe ratio  $S$  of the portfolio is the ratio of the expected excess return to the standard deviation. In terms of the zero-investment portfolio, the Sharpe ratio is<sup>5</sup>

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<sup>4</sup> The utility function used here can be any for which the standard problem has an “interior” solution. For state-independent utility, the efficient set is convex in a complete market; therefore,  $u(\cdot)$  can be taken to be the utility function of the representative investor. We express the budget constraint and hence the optimal portfolio in terms of the risk-neutral probabilities rather than the state prices for ease of comparison with later results. The risk-neutral probability of a state is equal to the state price multiplied by the risk-free discount factor,  $1 + r$ .

<sup>5</sup> The Sharpe ratio is generally defined using rates of return; however, since it is invariant to scale, it can also be defined in terms of dollar returns. Equation (4) and the following results give the true Sharpe ratio of the

$$S = \frac{\mathbb{E}[\tilde{x}]}{\sqrt{\text{Var}[\tilde{x}]}} = \frac{\sum p_i x_i}{\sqrt{\sum p_i x_i^2 - (\sum p_i x_i)^2}}. \quad (2)$$

Maximizing the Sharpe ratio is a standard quadratic optimization problem with linear constraints. Its solution is derived in the Appendix. The maximal-Sharpe-ratio portfolio (MSRP) with mean excess return  $\bar{x}$  has an excess return in state  $i$  of

$$x_i^* = \bar{x} + \frac{\bar{x}}{S_*^2} (1 - \hat{p}_i/p_i) \quad (3)$$

with an achieved maximal Sharpe ratio (MSR) of

$$S_*^2 = \mathbb{E}[(\hat{p}_i/p_i)^2] - \mathbb{E}^2[\hat{p}_i/p_i] \equiv \text{Var}[\hat{p}_i/p_i]. \quad (4)$$

As shown in (1) the likelihood ratio,  $\hat{p}_i/p_i$ , is proportional to the marginal utility in state  $i$ . Therefore, the variance of the likelihood ratio and the maximal Sharpe ratio will be larger when the representative investor is more risk averse or there is more risk in the economy so the variability of the argument of the utility function is greater.

The return on the MSRP in each state deviates from the expected return by an amount proportional to the difference of the probability likelihood ratio,  $\hat{p}_i/p_i$ , from unity. States with a risk-neutral probability exceeding their true probability will have smaller than average returns and vice versa. The larger the deviation between the risk-neutral and true probabilities, the greater is the difference from the mean return.

Comparing the MSRP to the solution to the standard problem in (1) we see that

$$x_i^* = \bar{x} - \frac{\bar{x}}{S_*^2} \left( \frac{u'(w_i^o)}{\mathbb{E}[u'(w^o)]} - 1 \right). \quad (5)$$

Since we have assumed complete markets, all optimal portfolios for state-independent utility functions are monotonically related to each other, and  $u(\cdot)$  can be any agent's utility. It is most convenient (but not necessary) to think of it as the utility function of the representative investor with  $w^o$  then being the market portfolio.

The excess return in state  $i$  on the MSRP differs from the average excess return by an amount proportional to the difference between the realized marginal utility in state  $i$  and the average marginal utility for any optimally invested portfolio. In particular,  $x^*$  is monotonically decreasing in the marginal utility. Since utility is concave,  $x^*$  is also monotonically increasing (but not linear) in  $w^o$ . For typical utility functions with  $u'''(\cdot) > 0$ ,  $x^*$  will be

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portfolio. In any period, the sample Sharpe ratio may differ from this population value. Its small sample properties will depend on the probability distribution of the states.

concave in  $w$ .<sup>6</sup> Generally, the total return including interest on the MSRP will exceed that on the optimal portfolio in the mid-portion of the outcomes and fall short of it for very good or very bad outcomes or, usually, both.

The highest excess return on the MSRP occurs in the state with the maximum market payoff. Depending on the probability distribution assumed, this may be infinite, but the highest return of the MSRP itself is bounded. Since the likelihood ratio,  $\hat{p}_i/p_i$ , is positive, the largest possible excess return is

$$x_{\max}^* \equiv \max_i x_i^* = \bar{x} + \frac{\bar{x}}{S_*^2} \left[ 1 - \min_i (\hat{p}_i/p_i) \right] \leq \bar{x} \left( 1 + 1/S_*^2 \right). \quad (6)$$

Whether or not this bound is achieved depends on the exact probability distribution and utility assumptions characterizing the market. Nevertheless, one important distribution-free property of the MSRP is that the upper tail is truncated.<sup>7</sup>

It should be no surprise that the MSRP has a bounded return. The Sharpe ratio comes from mean-variance analysis, which is justified for all probability distributions only for quadratic utility, and all quadratic utility functions have a satiation point. However, the portfolio's opposite tail is also peculiar. The MSRP need not have limited liability. In fact, the worst outcome is typically  $-\infty$ .

Since the excess returns are decreasing in the likelihood ratio,  $\hat{p}_i/p_i$ , the smallest excess return is

$$x_{\min}^* \equiv \min_i x_i^* = \bar{x} + \frac{\bar{x}}{S_*^2} \left[ 1 - \max_i (\hat{p}_i/p_i) \right]. \quad (7)$$

For many utility functions,  $u'(w) = \infty$  for  $w$ , the minimal value in the support of the utility function. So if the basis return is not bounded away from  $w$ ,  $\hat{p}_i/p_i \rightarrow -u'(w) = -\infty$ , and the lowest return realized on the MSRP will be  $-\infty$ . The explanation for this is that marginal utility is very high when the market return is at its worst. This makes the low-market-return state prices very high relative to their probabilities. Returns in these states can be “sold” for very high values and used to “purchase” returns in the less costly, but more likely middle range of outcomes. Therefore, the MSRP has worse performance than the market when the market return is very high or low; it outperforms the market only in the middle ranges.

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<sup>6</sup> For a representative utility function like  $u(w) = w - aw^\gamma$  with  $\gamma > 2$  and  $a > 0$ , the MSRP will be globally convex in the market. In this case using (5) and assuming differentiability, we have

$$\partial^2 x^*(w^o) / \partial w^2 = \bar{x} a (\gamma - 1)(\gamma - 2)(w^o)^{\gamma-1} / S_*^2 \mathbb{E}[u'(w^o)] > 0.$$

<sup>7</sup> In fact, as shown in a previous version of this paper, the Sharpe ratio can be improved in many cases by discarding high returns even if they cannot be sold. (This would be suboptimal in a complete market of course.) While we would not expect portfolio managers to actually discard any returns, certain institutional arrangements like the high-water mark performance fee on hedge funds may have a similar effect. See the discussion below.

Since  $\bar{x}$  can have any value,  $x_{\min}^*$  will exceed the risk-free return in absolute value for some choices of  $\bar{x}$  regardless of the probability distribution. In these cases, the MSRP will not have limited liability. However, if the number of states is finite or the probability likelihood ratio is otherwise bounded above across states, then the maximal Sharpe ratio can be achieved with a limited liability portfolio. Since  $x_{\min}^*$  is proportional to  $\bar{x}$ , by setting this mean excess return at a sufficiently low level, the smallest total return,  $1+r+x_{\min}^*$ , can be made positive and limited liability is achieved.

If there are infinitely many states and the probability likelihood ratio is unbounded, then a limited liability portfolio with a Sharpe ratio arbitrarily close to the maximal value can be formed by holding a portfolio with a total return of zero (i.e., excess return  $-(1+r)$ ) in the states with the highest likelihood ratios and excess returns proportional to those given in (3) for the states with lower ratios. Again by setting the mean excess return to a sufficiently small number, the fraction of states with a zero return can be made as small as desired and the resulting Sharpe ratio will be arbitrarily close to that achieved with an unconstrained portfolio.

## 2B. Sharpe Ratio in Mixed Jump Lognormal Diffusion Environment

This section analyzes the maximal-Sharpe-ratio problem in a continuous-state mixed jump diffusion environment. It is natural to use logarithmic returns as our basic elements since we want the returns per dollar to be independent and identically distributed over time. Furthermore, as we wish to consider returns over periods of varying durations, we start with a continuous-time model to enable scaling for any investment period. The reason for choosing the mixed diffusion jump process as our base case rather than the simpler pure diffusion is to examine both the case when the CAPM holds and does not hold in the limit. That is we want to allow, but not require, that the distribution converges to a diffusion with constant coefficients. The mixed diffusion jump distribution has all these properties.

We assume the market portfolio is a sufficient basis to describe the state space and let  $b$  denote the value on this basis. The basis evolves according to a mixed jump diffusion process; i.e.,  $d(\ln b) = \mu' dt + \sigma d\omega + \ln \tilde{\Gamma} d\phi$  where  $d\omega$  and  $d\phi$  are a Wiener process and Poisson process, respectively. The jump frequency of the Poisson process is  $\lambda$ ; the amplitude is  $\tilde{\Gamma}$ . The probability distribution of the jump amplitude is a realization of  $\Gamma_i$  with probability  $\pi_i$ .

### The Maximal Sharpe Ratio

As shown in equation (B9) in the Appendix, the maximal squared Sharpe ratio that can be achieved with a complete set of derivatives in this market is

$$S_*^2 = \exp[\rho^2 \sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2] - 1. \quad (8)$$

where  $\rho$  is the relative risk aversion of the representative investor. It is more common to think in terms of the risk-premium on the market than risk aversion. Since the representative investor holds the market basis, the market risk premium as given in (B11) is

$$\mu - r = \rho\sigma^2 + \lambda\sum_i\pi_i(1-\Gamma_i)(\Gamma_i^{-p}-1) \quad (9)$$

where  $\mu$  and  $r$  are the continuously compounded expected rate of return on the market and interest rate. For comparison, the squared Sharpe ratio of the market basis is from (B5)

$$S_b^2 = \frac{(1 - e^{-(\mu-r)T})^2}{\exp[\sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i - 1)^2] - 1} \quad (10)$$

In the no-jump case ( $\lambda = 0$ ), the two formulas simplify to<sup>8</sup>

$$S_*^2 = \exp\left[\frac{(\mu-r)^2}{\sigma^2} T\right] - 1 \quad S_b^2 = \frac{(1 - e^{-(\mu-r)T})^2}{e^{\sigma^2 T} - 1} \quad (11)$$

Table I shows the annualized Sharpe ratios,  $S/\sqrt{T}$  for the basis and for the MSRP for various parameter values.<sup>9</sup> The risk premiums used are 5%, 10%, and 15%. The logarithmic volatilities are 15%, 20%, and 25%. Two jump scenarios are considered: no jumps and jumps with an average frequency of once per year ( $\lambda = 1.0$ ). The jump sizes used were  $\Gamma_i - 1 = (5\%, -5\%, -10\%, -20\%)$  with probabilities  $\pi_i = (0.5, 0.4, .05, 0.05)$ .<sup>10</sup> This is consistent with market over the last twenty years. During 1984-2003, the S&P 500 changed by  $\pm 4.8\%$  or more on 20 days. A change of this size is 3.8 sigma event and would be expected to occur only once every 30 years in the absence of jumps rather than once every year as has happened. Half of these jumps were up and half were down; however, the lower tail was much longer with about one-tenth of the daily returns below  $-10\%$  and one-twentieth less than  $-20\%$ . No daily return exceeded 10% during those twenty years.

[Insert Table I here]

A more intuitive comparison of the two ratios is provided by the *apparent* alpha in the third sections of the two table portions. The apparent alpha is the extra average return that would have to be earned on the basis to achieve the same Sharpe ratio as the MSRP. To compute the apparent alpha, set the basis Sharpe ratio as given in (10) equal to the maximal Sharpe ratio and solve the “apparent” risk premium on the MSRP

<sup>8</sup> No properties of the continuous-time model were used to derive the results here. Of course, the continuous-time results can be derived directly by using the Black-Scholes model to determine the risk-neutral probabilities. This gives the formulas in (11) immediately.

<sup>9</sup> It is commonly assumed that the Sharpe ratio is proportional to the square root of the investment interval. However, this is precisely true only if the expected return and variance are proportional to the interval. Because it is the logarithmic variance and the continuously-compounded expected rate of return which are proportional to the interval, the index Sharpe ratio and maximal Sharpe ratio actually grow slower and faster than the square root of the time interval, respectively as shown in (B14).

<sup>10</sup> We also examined each of these jump sizes separately. Results were very similar in all non-zero jump cases and are available from the authors.

$$\mu_{\text{app}} - r = -\frac{1}{T} \ln \left( 1 - S_* \sqrt{e^{\sigma^2 T + \lambda T \Sigma_i \pi_i (\Gamma_i - 1)^2} - 1} \right). \quad (12)$$

The apparent alpha is then the difference between this risk premium and the actual risk premium. For example, in the no-jump case, if the basis risk premium is 10% and the volatility is 20%, the 13.1% improvement in the annual Sharpe ratio is apparently equivalent to an extra return of 139 basis points on the market basis. The other values for  $\mu_{\text{app}} - \mu$  are given in the table. Except for the Sharpe ratios measured over one month and with the smallest risk premium on the market, these “alphas” are economically meaningful numbers; however, we have been careful to label them apparent out-performance because there is no actual out-performance implied.

The possible improvement in the Sharpe ratio is greater the larger is the Sharpe ratio of the basis itself. So a higher risk premium or a smaller volatility on the basis allows for a greater percentage manipulation of the Sharpe ratio. Also a substantially greater improvement is possible over a one-year investment interval than over a monthly interval. This difference is also reduced for the monthly Sharpe ratios because the annualization chosen under-corrects the maximal Sharpe ratio and over-corrects the basis Sharpe ratio as shown in (B14). Of course, percentage improvements and the apparent alphas are not affected by the annualization.

In the no-jump case, the maximal and market Sharpe ratios measured with monthly data are very close in value. In fact could we observe the returns continuously, the basis and maximal Sharpe ratios would be equal. The limiting equality of these two measures can be seen in (B12) and is specific to the no-jump case. The lognormal model with no jumps and identically distributed returns converges to the continuous-time ICAPM of Merton (1973), where the continuous-time Sharpe ratio  $(\mu - r)/\sigma$  is the correct measure of portfolio performance. Of course for this limit to apply, both the investment period and the measurement period must become small together, and while the statistician controls the latter, he does not control the former.

In any event, even for the no-jump case, the ordinal properties are the same for all investment horizons. The MSR always exceeds the benchmark Sharpe over any finite interval so it would be judged a “better” portfolio.

In addition while shortening the measurement interval decreases this type-I error, it may well increase type-II error. We have assumed that managers have no special information and, therefore, that catching manipulation is the only goal. If some managers do or might have superior information, then we do want to be able to detect this as well. As always there will be a trade-off between the two types of error. In particular, if superior performance includes detecting temporarily mispriced assets, then the variance of returns will be less than proportional to the time interval as the pricing error will vanish on average over a longer interval. In this case, the best observation interval for detecting superior performance will depend on how quickly the market adjusts.<sup>11</sup>

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<sup>11</sup> Ferson and Siegel (2001) have examined the problem of maximizing the Sharpe ratio with superior information. In Appendix D we specialize their model to illustrate the horizon effect just discussed.

## The Maximal Sharpe Ratio Portfolio

We know that the MSRP is not optimal (except for quadratic utility). But what, other than maximizing the Sharpe ratio, are its properties? What is its return distribution? This subsection addresses this question. Since the realized return on the MSRP depends only on the market's return and not on whether or not any jumps have occurred, we confine our attention here to the no-jump case. The characteristics of the MSRP in the presence of jumps are very similar.

The excess return  $x^*(b)$  on the MSRP is

$$x^*(b) = \bar{x} + \frac{\bar{x}}{\mathcal{S}_*^2} \left( 1 - \exp \left[ \frac{1}{2} (\mu - r) \left( \frac{\mu + r}{\sigma^2} - 1 \right) T \right] b^{-(\mu - r)/\sigma^2} \right). \quad (13)$$

This payoff is illustrated in Figure 1 for  $\mu = 15\%$ ,  $r = 5\%$ ,  $\sigma = 15\%$ , and  $T = 1$ . The return shown has the same expected value as that on the basis, i.e.,  $\bar{x} = e^{\mu T} - e^{rT}$ . Note that the MSRP's return is substantially less than that on the basis in both tails. The MSRP's return exceeds that on the basis only over the range from about  $-4\%$  to  $25\%$ . Of course, this central range does account for nearly 60% of the probability distribution.

[Insert Figure 1 here]

Figure 1 also shows how the excess return on the MSRP compares to that on various optimal portfolios. The portfolios illustrated are optimal for investors with constant relative risk aversions of 1, 2, and 5.<sup>12</sup> The Sharpe ratios of these optimal portfolios are 0.397, 0.432, and 0.450. The maximal Sharpe ratio is, of course, larger than that on any of the optimal portfolios although the latter are, of course, the best ones for particular investors to hold.

The MSRP beats all of the optimal portfolios in the mid-ranges of their outcomes and falls short of them for extreme outcomes. The percentage of the time that the MSRP beats the optimal portfolios ranges from 57% for a relative risk aversion of one to 84% for a relative risk aversion of five. Nevertheless, the optimal portfolios are, of course, optimal and their performance in the tails of the distribution more than makes up for their deficiency in the midrange. (The basis portfolio is "near optimal" for a relative risk aversion of 2.3.)

Since the MSRP has the highest Sharpe ratio and beats each of the optimal portfolios a high percentage of the time, what, if anything, is so wrong with it? This question can be answered by examining the distribution of the portfolio returns. Because the basis,  $b$ , is lognormally distributed, any power of  $b$  is as well. So  $x^*(b)$  has a translated and reflected lognormal distribution.

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<sup>12</sup> The optimal portfolio for a constant relative risk aversion  $\rho$  is  $x_i = (1 + R_f) \left( (p_i / \hat{p}_i)^{1/\rho} \left[ \sum \hat{p}_i (p_i / \hat{p}_i)^{1/\rho} \right]^{-1} - 1 \right)$ .

$$p(x^*(b)) = \frac{\delta}{\bar{x}(1+1/S_*^2) - x^*} \frac{1}{\sigma\sqrt{T}} \phi\left(\frac{1}{\sigma\sqrt{T}} \left[-\delta \cdot \ln(1+S_*^2 - x^*/\bar{x}) - \frac{1}{2}(\mu-r)T\right]\right) \quad (14)$$

where  $\delta \equiv \frac{\sigma^2}{\mu-r}$ .

Figure 2 illustrates the probability distribution of the MSRP and compares it to the basis and various optimal portfolios. The basis and each of the optimal portfolios have a small right skewness. This is consistent with the empirical distribution of the market and the presumed decreasing absolute risk aversion of investors. The MSRP, on the other hand, has a reversed lognormal distribution bounded above by  $\bar{x}(1+1/S_*^2)$  and with extreme left skewness. Most investors would not pick the distribution offered by the MSRP over the others illustrated.

[Insert Figure 2 here]

Because both distributions are lognormal, all other moments are determined by the means and variances. In particular the normalized third and fourth moments are

$$\begin{aligned} \frac{\mathbb{E}[(b-\bar{b})^3]}{(\text{Var}[b])^{3/2}} &= (\omega_b + 2)\sqrt{\omega_b - 1} & \frac{\mathbb{E}[(b-\bar{b})^4]}{(\text{Var}[b])^2} &= \omega_b^4 + 2\omega_b^3 + 3\omega_b^2 - 3 \\ \frac{\mathbb{E}[(x^*-\bar{x}^*)^3]}{(\text{Var}[x^*])^{3/2}} &= -(\omega_x + 2)\sqrt{\omega_x - 1} & \frac{\mathbb{E}[(x^*-\bar{x}^*)^4]}{(\text{Var}[x^*])^2} &= \omega_x^4 + 2\omega_x^3 + 3\omega_x^2 - 3 \end{aligned} \quad (15)$$

where  $\omega_b \equiv e^{\sigma^2 T}$  and  $\omega_x \equiv e^{(\mu-r)^2 T / \sigma^2}$ .

Table II shows the skewness and kurtosis of the MSRP and the basis portfolio for volatilities of  $\sigma = 15\%$ ,  $20\%$ , and  $25\%$  and horizons of one month and one year.

[Insert Table II here]

There is some evidence that managed portfolios with high Sharpe measures are similarly negatively skewed and leptokurtic. Calamos (2003) reports the monthly statistics for four indices of convertible arbitrage hedge funds over a six to twelve year period ending in April 2000. The HFRI Convertible Arbitrage Index, the Hennessee HF Index, the CSFB/Tremont index and the EACM Relative-Value index had skewnesses of  $-1.52$ ,  $-1.23$ ,  $-1.66$  and  $-1.56$  and kurtosises (in excess of 3) of  $3.54$ ,  $3.17$ ,  $4.08$ , and  $4.46$ , respectively. These are more extremely left skewed and leptokurtic than the MSP portfolio as reported in Table II.

## 2C. Maximizing the Sharpe Ratio with Puts and Calls

In practice, a money manager may not be able to construct the Sharpe-ratio-maximizing portfolio because a complete market in contingent claims does not exist and it may be too expensive to create one with dynamic trading. However, even if the manager is allowed to trade only *one or two* ordinary put and call options, he can significantly enhance his Sharpe ratio. Furthermore, the best options to use will typically be liquid (near-the-money) options. Figure 1 indicates the best option strategy should probably include the sale of both out-of-the-money calls to remove the right tail and out-of-the-money puts to enhance the left tail of the distribution.

Suppose a money manager invests \$1 in the basis, sells  $\kappa$  European puts with a strike of  $K$  and sells  $\eta$  European calls with the strike of  $H$  ( $H > K$ ) to create the following simple piecewise linear return pattern

$$P = \begin{cases} b - \kappa(K - b) & b \leq K \\ b & K < b < H \\ b - \eta(b - H) & H \leq b. \end{cases} \quad (16)$$

Many interesting patterns are included here. For example, writing covered calls is equivalent to  $\kappa = 0$  and  $\eta = 1$ . Buying portfolio insurance is equivalent to  $\kappa = -1$  and  $\eta = 0$ . Partial write programs, partial insurance, and combinations are also included. The Sharpe ratio of the portfolio is

$$S = \frac{\mathbb{E}[P] - P_0 e^{rT}}{\sqrt{\mathbb{E}[P^2] - \mathbb{E}^2[P]}} \quad (17)$$

where  $P_0 = 1 - \kappa \mathcal{P}(1, T; K) - \eta \mathcal{C}(1, T; H)$  is the initial value of the portfolio, and  $\mathcal{P}$  and  $\mathcal{C}$  are the costs of the options. In our lognormal environment we will use the Black-Scholes model to price the options.

As shown in (C2) in the Appendix, the first and second moments are

$$\begin{aligned} \mathbb{E}[P] &= -\kappa K \Phi(-h_K^-) + (1 + \kappa) e^{\mu T} \cdot \Phi(-h_K) + e^{\mu T} [\Phi(h_K) - \Phi(h_H)] \\ &\quad + \eta H \Phi(h_H^-) + (1 - \eta) e^{\mu T} \Phi(h_H) \\ \mathbb{E}[P^2] &= \kappa^2 K^2 \Phi(-h_K^-) - 2\kappa(1 + \kappa) K e^{\mu T} \Phi(-h_K) + (1 + \kappa)^2 e^{(2\mu + \sigma^2)T} \Phi(-h_K^+) \\ &\quad + e^{(2\mu + \sigma^2)T} [\Phi(h_K^+) - \Phi(h_H^+)] + \eta^2 H^2 \Phi(h_H^-) \\ &\quad + 2\eta(1 - \eta) H e^{\mu T} \Phi(h_H) + (1 - \eta)^2 e^{(2\mu + \sigma^2)T} \Phi(h_H^+) \end{aligned} \quad (18)$$

$$\text{where } h_z \equiv \frac{-\ln Z + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad h_z^\pm \equiv h_z \pm \sigma\sqrt{T}.$$

The Sharpe ratios for portfolios holding just one option are given by setting  $\kappa$  or  $\eta$  to zero. The Sharpe ratios for portfolios holding options at more than two strikes can be computed similarly. However, only one or two options are required to achieve most of the possible improvement in the Sharpe ratio.

For example for a basis with  $\mu = 15\%$  and  $\sigma = 15\%$  and interest rate of 5%, the Sharpe ratio is 0.631 over a one-year horizon. By selling 0.843 calls at a strike of 1.0098 the Sharpe ratio can be pushed to 0.731.<sup>13</sup> Using two options allows an improvement of the Sharpe ratio to 0.743. The best two-option portfolio is characterized by  $\kappa = 2.58$ ,  $K = 0.88$ ,  $\eta = 0.77$ ,  $H = 1.12$ . The maximal Sharpe ratio is 0.748, so 86% of the total possible increase in the Sharpe ratio can be achieved with one option contract, and 96% can be achieved with just two option contracts.

The improvement in the Sharpe ratio is not critically sensitive to the exact value of the strike price. For example, a Sharpe ratio of 0.716 or 0.694 can be achieved by using a call that is 5% in-or out-of-the-money in place of the best single call with a strike 1% in-the-money. A Sharpe ratio of 0.737 can be achieved using both of these options. Near-the-money options are very liquid, and seldom is the strike price gap as large as 10%. Therefore, simple put and call strategies should be able to provide most of the improvement possible in the Sharpe ratio.

Figure 3 plots the payoff on the put-call-stock portfolio and compares it to that on the MSRP.<sup>14</sup> The distributions are similar in many respects though the returns on the option portfolio are larger for both very high and low returns on the basis. This means that the option portfolio has less negative skewness than the MSRP. It may have more or less kurtosis than the MSRP. The return is no longer bounded above, but it still is less than –100% in the worst cases.

[Insert Figure 3 here]

## 2D. The Maximal Sharpe Ratio for Other Distributions

The equation for the maximal Sharpe ratio and the MSRP are completely general; however, many of the specific results in the previous section depend on the lognormality assumption. In this section we briefly illustrate results for two other cases, the normal distribution and bilinear utility.

The normal distribution is the one usually assumed to justify mean-variance analysis, but even with a normally distributed benchmark, the Sharpe ratio is subject to manipulation. The likelihood ratio can be determined as in (1) from the utility function of the representative

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<sup>13</sup> By put-call parity, selling 5.36 puts at the same strike gives identical results.

<sup>14</sup> The illustrated best put and call portfolio does not appear to be a “best” fit for the curve describing the maximal-Sharpe-ratio portfolio for the simple reason it is not best fit for the illustrated curve.

investor who holds the market. The typical conjugate assumption for a normal distribution is exponential utility. In this case the probability likelihood ratio is<sup>15</sup>

$$\frac{\hat{p}(b)}{p(b)} = \frac{u'(b)}{\mathbb{E}[u'(b)]} = \exp\left[-\frac{\bar{b} - R_f}{\sigma_b^2}(b-1) + \frac{\bar{b}^2 - R_f^2}{2\sigma_b^2}\right]. \quad (19)$$

where  $\bar{b}$  and  $\sigma_b$  are the mean and standard deviation of the benchmark portfolio's rate of return.

The maximal Sharpe ratio is<sup>16</sup>

$$S_* = \sqrt{\exp\left[(\bar{b} - R_f)^2 / \sigma_b^2\right] - 1} = \sqrt{\exp(S_b^2) - 1} \quad (20)$$

where  $S_b$  is the Sharpe ratio of the benchmark basis. The maximal Sharpe ratio clearly exceeds the benchmark's Sharpe ratio, and the larger is the market's Sharpe ratio, the larger is this difference. For example, the annual Sharpe ratio of the S&P 500 index was 0.450 from 1926–2000. Assuming normality, the maximal Sharpe ratio would be 0.474, which is 5% higher. From 1990–2000, the S&P Sharpe was 0.655. Again assuming normality, the maximal Sharpe ratio would be 0.732, almost 12% higher.

The return on the Sharpe-ratio-maximizing portfolio deviates substantially from normal. The return is

$$x^*(b) = \bar{x} \left\{ 1 + S_*^{-2} \left[ 1 - \exp\left(-\frac{\bar{b} - R_f}{\sigma_b^2}(b-1) + \frac{\bar{b}^2 - R_f^2}{2\sigma_b^2}\right) \right] \right\}. \quad (21)$$

This is increasing in the basis, but it is bounded above by  $x_{\max}^* = \bar{x} [1 - e^{-(\bar{b} - R_f)^2 / \sigma_b^2}]^{-1}$  and unbounded below. As in the lognormal case, the MSRP's return has a "reflected" lognormal distribution; i.e.,  $x_{\max}^* - x^*$  is lognormally distributed.

The representative utility function is equally, if not more important in determining the MSRP. For example, as an extreme case, suppose the utility function of the representative investor has the bilinear form with two different regions of linear utility. In this case, regardless of the probability distribution of the states, the MSRP only has two distinct excess returns, negative for low market outcomes and positive for high outcomes.<sup>17</sup>

<sup>15</sup> For exponential utility,  $u(w) = -e^{-aw}$ , the representative investor will hold the benchmark portfolio unlevered for a risk aversion parameter of  $a = (\bar{b} - R_f) / \sigma_b^2$ . For this investor, expected marginal utility is  $\mathbb{E}[u'(b)] = \exp\left(-(\bar{b} - R_f)[1 + \frac{1}{2}(\bar{b} + R_f)] / \sigma_b^{2a}\right)$ .

<sup>16</sup> Recall that if  $z$  is normally distributed with mean  $\bar{z}$  and standard deviation  $\sigma_z$ , then  $\mathbb{E}[e^{\alpha z}] = \exp(\alpha \bar{z} + \frac{1}{2} \alpha^2 \sigma_z^2)$ . The result in (20) is similar to that in (11), but recall that  $(\mu - r) / \sigma$  was not the benchmark's Sharpe ratio in the lognormal economy.

<sup>17</sup> If the marginal utility is 1 for  $b < \underline{b}$  and  $\alpha$  for  $b > \underline{b}$ , the maximal-Sharpe-ratio portfolio has only two distinct excess return outcomes of  $-\alpha \bar{x} / [P(1 - \alpha)]$  and  $\bar{x} / [(1 - P)(1 - \alpha)]$  where  $\underline{P} \equiv \text{Prob}\{b \leq \underline{b}\}$ . The Sharpe ratio is  $\sqrt{\underline{P}(1 - \underline{P})(1 - \alpha) / [\underline{P} + \alpha(1 - \underline{P})]}$ .

### 3. Sharpe Maximization with a Changing Opportunity Set

When considering the ways in which any measure can potentially be manipulated, intertemporal issues should not be ignored. The Sharpe ratio, like virtually every other performance measure, is predicated on identically-distributed returns since each return is included in the overall score in the same fashion. In an environment in which it were known that returns at different times had different distributions, a proper measure of performance should take this into consideration; nevertheless, any such differences are commonly ignored in practice.

Furthermore, even if all returns are *ex ante* identically distributed, the *ex post* realized returns will typically differ from the true distribution. Therefore, a money manager who has been particularly lucky or unlucky knows that his return distribution in the future will likely not be similar to what he has experienced. Consequently, to maximize the Sharpe ratio, he should modify his portfolio to take into account the difference in the distributions between the realized and future returns just as if the returns distribution was not *ex ante* identical each period.

To address this problem in a general framework, we now assume that there are different distributions of excess returns available at different times. These distributions are indexed by the subscript  $k$ . We maintain the assumption that the returns in different periods are independent. The money manager knows from which distribution each period's excess return will be drawn before forming that period's portfolio. The portfolio selected by the manager will have different properties depending on the prevailing distribution of returns, but the calculation of the Sharpe ratio does not take these differences into consideration; it is computed from the overall mean and variance.<sup>18</sup>

Since returns are independent, the manager will select the same portfolio in each period with the same prevailing distribution, and the overall Sharpe ratio will depend only on the means and variances of these portfolios constructed for the individual distributions and the frequencies of the distributions. Let  $\mu_k$  and  $\sigma_k^2$  denote the mean excess return and variance of the portfolio associated with distribution  $k$ , and let  $\pi_k$  denote the frequency of occurrence of distribution type  $k$ . The overall mean excess return and variance on the portfolio are

$$\begin{aligned}\bar{\mu} &= \sum \pi_k \mu_k \\ \sigma^2 &= \sum \pi_k (\sigma_k^2 + \mu_k^2) - \bar{\mu}^2 = \sum \pi_k \sigma_k^2 + \sum \pi_k (\mu_k - \bar{\mu})^2.\end{aligned}\tag{22}$$

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<sup>18</sup> One maintained assumption of this paper is that the manager has no special information. This can be true here as well. All market participants may also know from which distribution the return is drawn. We only assume that the statistician is unaware of the prevailing return distribution or ignores this information in constructing a single Sharpe ratio. Alternatively, the manager's information may be private provided it is not so good as to create an arbitrage opportunity. In this case, the Sharpe ratio is computed from the manager's probability distribution. This will also be the true Sharpe ratio if the manager's information is correct.

The second statement of variance shows that the Sharpe ratio will be penalized not only for the variance of each portfolio but also for any difference in means among the portfolios. Since leverage can be used to set each portfolio's mean excess return while maintaining the same Sharpe ratio, it might seem that the various means could be kept equal to avoid this penalty. While this is true, it is not the Sharpe-ratio-maximizing solution. Fixing the different portfolio means to be equal also constrains the standard deviations because each can be no smaller than the mean divided by the maximal within-distribution Sharpe ratio. This constraint has its own penalty and the Sharpe-ratio-maximizing solution trades off the two penalties.

The Sharpe-ratio-maximizing solution can be most easily seen in the following relation using the individual portfolio excess means and Sharpe ratios,  $S_k = \mu_k / \sigma_k$ . The overall Sharpe ratio is

$$S = \frac{\bar{\mu}}{\sqrt{\sum \pi_k (\sigma_k^2 + \mu_k^2) - \bar{\mu}^2}} = \frac{\bar{\mu}}{\sqrt{\sum \pi_k \mu_k^2 (1 + 1/S_k^2) - \bar{\mu}^2}}. \quad (23)$$

The individual portfolio excess means,  $\mu_k$ , can be set at any positive level without altering the individual Sharpe ratios. So by examination of (23), it is clear that the Sharpe-ratio-maximizing strategy involves holding for each of the  $k$  portfolios, one that maximizes the individual Sharpe ratio for the  $k^{\text{th}}$  distribution. The first-order condition for the individual portfolio means are<sup>19</sup>

$$\mu_k^* = \bar{\mu} \Gamma / (1 + S_{k*}^{-2}) \quad \text{where} \quad \Gamma \equiv \left[ \sum \pi_k / (1 + S_{k*}^{-2}) \right]^{-1}. \quad (24)$$

As can be seen the means are not equal. The Sharpe-ratio-maximizing solution uses more leverage, thereby creating a higher mean, for those distributions that permit a larger individual-portfolio Sharpe ratio. Any two distributions that permit the same maximal Sharpe ratio will have the same mean regardless of any differences in the distributions.

The overall maximum Sharpe ratio that can be achieved is

$$S_* = 1 / \sqrt{\Gamma - 1}. \quad (25)$$

For example, if the market return is lognormal with a logarithmic standard deviation of 20% and a mean excess rate of return of either 5% or 15%, then the Sharpe ratios for the two distributions' portfolios are 0.254 and 0.869. If the two means are equally likely, the overall Sharpe ratio is 0.570, which is above the average of 0.560. However, the other case is also

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<sup>19</sup> As always, the overall portfolio mean,  $\bar{\mu}$ , can be set at any positive level without affecting the Sharpe ratio. The Sharpe ratio in (23) can be maximized by minimizing the first term in the denominator,  $\sum \pi_k \mu_k^2 (1 + 1/S_k^2)$ , subject to the constraint  $\bar{\mu} = \sum \pi_k \mu_k$ .

possible, the overall Sharpe might be smaller than the average of the individual Sharpe ratios.<sup>20</sup>

As discussed at the start of this section, dynamic Sharpe-maximization strategies can also be analyzed in a similar fashion. Consider a money manager who has an existing history of returns and wishes to maximize the Sharpe ratio computed from these and future returns. Suppose the realized distribution of returns differs from those available in the future because the manager was lucky or unlucky. Then the manager is effectively choosing portfolios from two different distributions — those of the past and future. At this point the manager cannot alter the past portfolio; however, its performance will affect his choice for the future.

The Sharpe ratio over the entire period will depend on the mean and variance of both the history and the future. By analogy with (23), the Sharpe ratio over the entire measurement period will be

$$S = \frac{\alpha\mu_h + (1-\alpha)\mu_f}{\sqrt{\alpha\mu_h^2(1+1/S_h^2) + (1-\alpha)\mu_f^2(1+1/S_f^2) - [\alpha\mu_h + (1-\alpha)\mu_f]^2}}. \quad (26)$$

where  $\alpha$  is the fraction of the whole period that has passed, and subscripts  $h$  and  $f$  refer to the history and the future.

As before, the overall Sharpe ratio is maximized by choosing a portfolio in the future that maximizes the future Sharpe ratio, i.e.,  $S_f = S_*$ . The mean excess return in the future that maximizes the overall Sharpe ratio is<sup>21</sup>

$$\mu_f^* = \begin{cases} \mu_h(1+1/S_h^2)/(1+1/S_*^2) & \text{for } \mu_h > 0 \\ \infty & \text{for } \mu_h \leq 0. \end{cases} \quad (27)$$

If the manager has been lucky in the past and  $S_h > S_*$ , then the portfolio should be targeted at a lower mean excess return and lower variance in the future. This allows the past good fortune to weigh more heavily in the overall measure.<sup>22</sup> Conversely, if the manager has

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<sup>20</sup> The overall Sharpe ratio can also be expressed as  $S_*^2/(1+S_*^2) = \sum \pi_k S_{k*}^2 / (1+S_{k*}^2)$ . The function  $z^2/(1+z^2)$  is S-shaped, convex below  $z = 1/\sqrt{3} \approx 0.577$  and concave above. Therefore, by Jensen's inequality, the maximized Sharpe ratio will be above the average of the Sharpe ratios if they are typically smaller than this, but less than the average if they are large.

<sup>21</sup> In our previous analysis we did not have to consider the case of  $\mu_k \leq 0$ . We ignored luck and used the true *ex ante* distributions for evaluation. The *ex ante* mean of a maximal-Sharpe-ratio portfolio must be positive since the portfolios  $x_i$  and  $-x_i$  have the same variance and opposite means. Here we ignore luck only insofar as the future returns are considered.

<sup>22</sup> For a fixed Sharpe ratio, the overall mean is linear in the future leverage while the overall standard deviation is convex. The proportional changes in the mean and standard deviation with respect to future leverage are equal when the future and historical Sharpe ratios are equal. Therefore, when the historical Sharpe ratio is less than the future Sharpe ratio, increasing leverage increases the overall mean at a faster rate than the standard deviation and vice versa.

been unlucky and  $S_h < S_*$ , then in the future, the portfolio should be targeted at a higher mean excess return than that so far realized and a higher variance. In the extreme, if the average excess return realized has been negative, then the manager should use as much leverage as possible (ideally infinite) to minimize the impact of the poor history.

The Sharpe ratio that can be achieved over the entire period is<sup>23</sup>

$$S = \begin{cases} \sqrt{\frac{S_*^2 S_h^2 + \alpha S_h^2 + (1-\alpha) S_*^2}{1 + (1-\alpha) S_h^2 + \alpha S_*^2}} & \text{for } S_h > 0 \\ S_* \sqrt{\frac{1-\alpha}{1+\alpha S_*^2}} & \text{for } S_h \leq 0. \end{cases} \quad (28)$$

Figure 4 plots the overall Sharpe ratio as a function of the realized historical Sharpe ratio for histories of different durations. The overall Sharpe ratio can be maintained above (is forced below)  $S_*$  whenever the past performance has been good (bad). The realized Sharpe ratio, of course, has the most impact when the past history is long ( $\alpha \approx 1$ ).

[Insert Figure 4 here]

As can be seen in the graph or from equation (28), the over-all Sharpe ratio is increasing and convex in the historical Sharpe ratio. Therefore, dynamic Sharpe maximizing strategies should, on average, be able to produce a Sharpe ratio higher than  $S_*$ . Brown, et al (2004) provide some evidence that some Australian money managers engage in a pattern of trading consistent with the behavior described here.

## 4. A Manipulation-Free Measure

Thus far we have shown that the Sharpe ratio can be manipulated by a manager who has no special information. An obvious question is whether it is possible to derive a performance measure that cannot be manipulated. This section examines that question and shows when such a measure exists and characterizes it.

An important first question is what does it mean for a measure to be manipulation-free. What exactly should it encourage the manager to do and not do? Intuitively, if a manager has no private information and markets are efficient, then holding the market portfolio or some other appropriate benchmark should maximize the measure's expected value. By this definition manipulation is the rebalancing of the portfolio away from the benchmark even when there exists no informational reason to do so. Thus, a manipulation-free measure should satisfy the following conditions:

First, conditional upon having no special skill or information, the expected value of the proposed performance measure should be unconditional on the history of past returns.

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<sup>23</sup> Because the optimal future leverage is infinite when  $\mu_h \leq 0$ , the past Sharpe ratio does not affect the overall Sharpe ratio and the value of the overall Sharpe ratio for  $S_h < 0$  is the same as for  $S_h = 0$ .

The dynamic re-balancing example above shows that the Sharpe ratio may be easily gamed if the past track record is allowed to determine the measure's value. This extends to option-like payoffs, since they have a dynamic rebalancing representation. The intuition behind this restriction is that the manipulation of the Sharpe ratio is achieved by shifting payoffs through time or across states. Since each shift is done at the correct risk-neutral prices, the effect is to costlessly increase the performance measure.

As the analysis shows below, this condition essentially restricts measures to the set of concave increasing utility functions. Unlike other potential functions, these provide orderings that penalize portfolio changes that result only in a mean-preserving or mean-reducing spreads. This sharply limits a manager's ability to increase his score by simply buying and selling options, or by changing the portfolio depending upon past performance in the measurement period. The fact that this must hold given any history further restricts our attention to time-separable utility specifications. Otherwise, one cannot guarantee that past returns will not influence the measure-maximizing portfolio in the current period.

The second criterion is that the measure should not depend on the initial value of the portfolio. This essentially creates a "level playing field" in that portfolios that begin with either large or small sums of money are treated the same. This seemingly mild restriction is actually quite powerful because among all utility functions only the power utility function has this property.

The third criterion is that the market portfolio should be optimal for an investor without private information and that the measure should select that as the benchmark. This essentially ensures that those without superior information will wish to hold the market portfolio. As a practical matter this criterion simply pins down the measure's "risk aversion" coefficient, making the proposed measure unique. This restriction may be relaxed to allow for a range of preferences if desirable, however.

#### 4A. The Manipulation-Free Measure: A Formal Derivation

A performance measure, like the Sharpe ratio, is an ordinal function,  $\Theta(\cdot)$ , of the returns.<sup>24</sup> To have a manipulation-free measure, specific outcomes for some components of the vector of returns across states must not alter the rankings of other components. Precisely, let  $\mathbf{r}$  and  $\mathbf{q}$  be two subsets of returns across states. Then the measure is manipulation free if  $\Theta((\mathbf{r}_1, \mathbf{q}_0)) > \Theta((\mathbf{r}_2, \mathbf{q}_0))$  implies  $\Theta((\mathbf{r}_1, \mathbf{q})) > \Theta((\mathbf{r}_2, \mathbf{q}))$  for all  $\mathbf{q}$ .<sup>25</sup> This structure is well-known in multi-attributed utility theory, and the desired property is called strong utility independence.

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<sup>24</sup> We confine our attention to one-dimensional measurement scores that give a complete ranking among portfolios. Manipulation-free measurement scores that assign a vector of numbers to each portfolio can be easily constructed. One trivial example is a vector of all the portfolio's returns. However, such a measure hardly provides a summary statistic.

<sup>25</sup> If we interpret the component,  $\mathbf{q}$ , as a null vector, the same argument ensures that we can compute for example a one, five and ten-year Sharpe ratio from the same data, and that each of them is meaningful.

A necessary and sufficient condition for utility independence is that the utility function has an additive representation in the separate attributes. Therefore, a manipulation-free performance measure must have a representation as<sup>26</sup>

$$\Theta(\mathbf{r}) = \sum \theta_i(r_i). \quad (29)$$

In practical applications, of course, a performance measure is a function not of the state-by-state returns, but of the realized excess returns over time, and each return is treated in an identical fashion. That is, the sample performance measure is  $\hat{\Theta}(\mathbf{r}) = \sum_i \theta(r_i)$  where each period's realization represents an outcome state. Assuming the realizations of the states over time are iid, then the process is ergodic, the frequencies of the states will be equal to their *ex ante* probabilities giving  $\hat{\Theta}(\mathbf{r}) = \sum_i \theta(r_i) = \sum_i p_i \theta(r_i) = \mathbb{E}[\theta(\tilde{r})]$ .<sup>27</sup>

This analysis implies that portfolios should be evaluated by computing the expectation of some function of returns. That is, we should compute expected utility under the reasonable assumptions that  $\theta(\cdot)$  is increasing and concave.<sup>28</sup> The only remaining question is what function should be used.

Having reduced a multi-period evaluation effectively to a single-period expected utility assessment, it is logical to restrict  $\theta(\cdot)$  to be consistent with utility functions that are myopic in a static world, i.e., the linear risk tolerance class. Furthermore, since the measure considers only returns and not the level of wealth, and since we require its value to be independent of the portfolio's initial value, we may restrict our attention to power utility. This leads to the following performance score

$$\Theta \equiv (1 + R_f)^{-1} \left( \mathbb{E} \left[ (1 + R_f + \tilde{x})^{1-\rho} \right] \right)^{1/(1-\rho)}. \quad (30)$$

where  $\tilde{x}$  is the excess return on the portfolio. In practice, the average is used in place of the expectation. This is a monotone transformation normalized so the risk-free asset has a score of unity. We choose this transformation since  $\Theta$  can be interpreted as the proportional certainty equivalent excess return.<sup>29</sup>

<sup>26</sup> See, for example, Debreu (1960).

<sup>27</sup> This same result is true even if the sample measure is based on some time-weighting of the past returns  $\hat{\Theta}(\mathbf{r}) = \sum_i w_i \theta(r_i)$  where the weights sum to one. In addition, if the manager is concerned about the small sample properties of his performance score and maximizes some expected utility from the recorded measure,  $\mathbb{E}[u(\Theta(\tilde{r}))]$ , the same result holds using  $u(\Theta(\cdot))$  as the relevant performance measure in place of  $\Theta(\cdot)$ .

<sup>28</sup> The manipulation-free property does not ensure the measure will be a good gauge of performance. We would certainly want to impose the additional condition that  $\theta(\cdot)$  be increasing so that higher returns were ranked better than lower returns. We also want  $\theta(\cdot)$  to be concave. This is consistent with risk aversion and also ensures that the measurement score is meaningful. If  $\theta(\cdot)$  were convex, the score could be made unboundedly large.

<sup>29</sup> We define the proportional certainty equivalent excess return for the utility function  $u(\cdot)$  as  $u(\Theta \cdot (1 + R_f)) = \mathbb{E}[u(1 + R_f + \tilde{x})]$ . The proportional certainty equivalent excess return given in (30) is that for the power utility function with constant relative risk aversion of  $\rho$ ,  $u(z) = z^{1-\rho}/(1-\rho)$

#### 4C. Discussion of the Manipulation-Free Measure

The manipulation-free measure  $\Theta$  is valid for any value of  $\rho \geq 0$ . One further criterion can be used to specify the appropriate value or a range of values. If the CAPM holds or more generally if markets are complete, then, in the absence of any special information, the market portfolio should be efficient (i.e., optimal for the representative investor). That is, we want the market to have a  $\Theta$  measure as large as that for any other portfolio constructed without special information. We can assure this by setting  $\rho$  to the relative risk aversion of the representative investor. In practice, we can set  $\rho = (\mu_m - r) / \sigma_m^2$  where  $\mu_m$  and  $\sigma_m^2$  are the continuously-compounded expected rate of return and logarithmic variance of the market portfolio and  $r$  is the continuously-compounded interest rate.<sup>30</sup> Historically this number is about 2.

Two special cases of the manipulation-free measure are  $\rho = 0$  and  $\rho = 1$ . Using the former corresponds to ranking portfolios on their arithmetic average return ignoring risk. Using  $\rho = 1$ , more precisely,  $\Theta \equiv (1 + R_f)^{-1} \exp(\mathbb{E}[\ell n(1 + R_f + \tilde{x})])$ , corresponds to ranking portfolios on their geometric mean. On any sample, this ranks portfolios on their total growth in value over the investment period per dollar originally invested

In theory, the MSRP will score very badly by this measure. For example, in the lognormal case the  $\Theta$  measure for the MSRP will be the minimum possible value (i.e., 0 for  $\rho < 1$  and  $-\infty$  for  $\rho \geq 1$ ) as it does not have limited liability. However, the actual measured performance depends on the small sample properties of the statistic. For the market's mean and variance, a bankruptcy event only occurs about once every thousand years with annual rebalancing or once every  $10^{14}$  years with monthly rebalancing so in practice, the sample mean of the MSRP's measurement score will almost always exceed the minimum possible value and thus biased high. Furthermore, since the MSRP beats the market benchmark throughout the midrange of returns, the sample  $\Theta$  measure for the MSRP will may exceed the market's during periods with low realized volatility.

To get a sense of the small sample properties of  $\Theta$ , we used numerical calculations assuming a lognormal market with an 8% annual risk premium and a 20% annual volatility show that the MSRP has a higher Sharpe ratio than the market 63% of the time using 10 years of quarterly observations. However, the MSRP has a lower  $\Theta$  measure 56% of the time for  $\rho = 2$  and 75% of the time for  $\rho = 4$ . The next section compares how the Sharpe and  $\Theta$  measures perform based on the actual returns of mutual funds and hedge funds.

#### 4D. Sharpe Ratios vs. the Manipulation-Free Measure: Ranking Funds

In this section we apply the proposed  $\Theta$  measure to fund data. Using monthly returns from both the hedge fund universe and the mutual fund universe, we examine the conditions

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<sup>30</sup> This value is exactly correct if the market portfolio is the true portfolio of all wealth and it has a lognormal distribution. If the measure is used just on portfolios of financial assets, a value somewhat larger than this should probably be used. For example, if the equity market portfolio is uncorrelated with other assets, then  $\rho = (\mu_m - r) / (w_m \sigma_m^2)$  where  $w_m$  is the fraction of all wealth invested in equities. As shown below, the ranking of portfolios is not too sensitive to this value.

under which  $\Theta$  provides different performance rankings from the traditional Sharpe ratio. Of particular interest is whether the new utility-based measure ranks funds with negative skewness lower than does the Sharpe ratio. One important issue in the practical application of  $\Theta$  is the choice of the parameter  $\rho$ , which must be set by the investor. If the rankings are extremely sensitive to “reasonable values” of  $\rho$ , then that makes the measure’s use problematic.

### Data and Methodology

For the mutual funds, we use return data from the CRSP mutual fund files over the period October 1993 through September 2003. For the hedge fund analysis, we use the TASS hedge fund database over the period October 1992 through September 2002. We split both samples into two periods in order to examine the stability of  $\Theta$ . For the mutual fund data, we further restrict our attention to equity funds with the ICDI objectives of Aggressive Growth, Global Equity, Growth and Income, International Equity, Income, Long-Term Growth, Precious Metals, Sector Funds, and Utilities. For hedge funds we use all available funds in the universe over each time period.

For each fund we use monthly returns to calculate the Sharpe ratio and the utility-based measure  $\Theta$  for four values of  $\rho$ , 0, 1, 2, and 6. As noted above,  $\rho = 0$  is a ranking based on arithmetic average alone, while  $\rho = 1$  is a ranking on geometric return (total growth per dollar invested). The former ignores risk in the traditional sense. The latter ignores risk *ex post* focusing only on the total return although for funds with the same expected return, it would typically be lower for more volatile returns. A value of  $\rho = 2$  is typical for risk-aversion as exhibited in the market implying (at least in a lognormal market) a risk premium equal to twice the variance. Although this is a standard assumption, the literature on the equity premium is far from reaching a consensus. We therefore also consider a higher value,  $\rho = 6$ , as a form of model calibration. These results do differ somewhat from those for smaller risk aversions as might be expected.

### Rank Correlation Results

Panel 1 in Table III shows the rank correlation between the Sharpe ratio and the utility-based measure for hedge funds and for various types of equity mutual funds over the first sample period. The rank correlations are close to one for most styles of mutual funds and for most ranges of the risk-aversion parameter  $\rho$ . Across all funds, the rank correlation ranges from 0.951 for  $\rho = 0$  to 0.966 for  $\rho = 6$ . For  $\rho = 2$ , the value for the sample period is 0.986 across all mutual funds. The fact that the rank correlations are relatively consistent for values of  $\rho$  between 0 and 6 implies that the measure is not too sensitive in this regard.

Because both the manipulation-free measure and the Sharpe ratio are designed to rank performance for funds holding an investor’s entire wealth, it is not logically consistent to apply them to under-diversified portfolios. Nevertheless, Sharpe ratios have commonly been used in just this way. Therefore, we also report in Table III rank correlations for subcategories of mutual funds within major style groups. The rank correlation between the Sharpe ratio and the manipulation-free measure with  $\rho = 2$  ranges from a high of 0.983 for the international equities category to a low of 0.695 for the precious metals category. While this variation may due to the small sample size of the precious metals group, note that utility

funds have the same sample size and a much higher rank correlation. Another explanation for the low precious metals correlation coefficient may be that the relative volatility of the asset class draws the Sharpe ratio away from the manipulation free measure more often.

For hedge funds over approximately the same period the Sharpe ratio and  $\Theta$  disagree to a larger extent about relative fund rankings. Using data from October 1992 through September 1997 rank correlations range from 0.595 for  $\rho = 0$  to 0.895 for  $\rho = 6$ , with a value of 0.881 for  $\rho = 2$ . In other words the Sharpe ratio does a poorer job at ranking hedge funds according to a utility-based measure over this sample period.

Panel 2 in Table 3 reports results for the second sample period. The overall results for mutual funds are slightly lower for  $\rho = 2$ . The rank correlation between the manipulation-free statistic and the Sharpe ratio is 0.886 over the period October, 1998 through September, 2003. This is substantially lower than the first period rank correlation. On the other hand, the rank correlation between the Sharpe ratio and  $\Theta$  with  $\rho = 2$  for hedge funds in the second period increases substantially to 0.881, almost the same value as for the mutual funds. Thus, it appears that there is significant intertemporal variation in the degree to which Sharpe ratio rankings for investment funds are consistent with a utility-based measure. Although it never does a terrible job, the Sharpe ratio works better in some periods than in others.

### Skewness Results

A major implication of the analysis in the first part of this paper is that the Sharpe ratio can be manipulated through the use of option-like payoffs that create a fund return distribution that has negative skewness. In this section we test whether the deviations in the rank-orderings of funds under the Sharpe ratio and the manipulation-free measure can be explained by negative skewness. The approach is straightforward. We compute the difference in the rank percentile under the two measures for each fund and then regress this difference on the fund's normalized skewness. The normalized return skewness is the third moment of the monthly return distribution divided by the cube of the standard deviation. Let the Sharpe ratio for fund  $i$  be  $S_i$ .  $R$  denotes percentile ranking of the fund in its universe from 0 to 1. We estimate:

$$R(S_i) - R(\Theta_{2,i}) = \alpha + \beta \text{Skewness}_i + \epsilon_i \quad (34)$$

In the above formulation, if negative skewness is indeed associated with a higher Sharpe ratio ranking compared to the rank under the  $\Theta$  measure, the regression coefficient,  $\beta$ , will be negative and significant. The results of this regression for each sample period are in Table IV. For the universe of mutual funds, we find a significant negative relationship between skewness and positive bias in the ranking under the Sharpe ratio for the first period, and a positive and significant relationship in the second period. This flip in sign is not, however, consistent across most fund styles. The results for various categories in the two periods are mixed. Thus, at least for mutual funds, it is difficult to draw any inference regarding evidence for or against Sharpe ratio improvement driven by intentional or unintentional manipulation of the third moment of returns.

The evidence for hedge funds is different, however. The coefficient for  $\Theta$  with  $\rho = 2$  is significant and negative for both time periods. This indicates that a positive divergence in ranking under the Sharpe ratio and  $\Theta$  measure is related to increased negative skewness in

fund returns over both time periods. This, in turn, is consistent with the possibility that at least some hedge funds in the sample are effectively (or actually) selling out-of-the-money options. This conclusion for hedge funds contrasts with the mutual fund results over the two sample periods. It seems extremely unlikely that as a group mutual funds were selling out-of-the-money options in the first sample period and buying out of the money options in the second period. More likely there was either some secular change in the skewness of the market (i.e. the common factor across equity funds) or a major change in the distributional characteristics of the equities held in the typical mutual fund portfolio.

Our empirical analysis suggests why the Sharpe ratio has enjoyed a long history of use; it frequently works well. When benchmarked against a utility-based performance measure constructed so as to be consistent with the CAPM, the Sharpe ratio has a high rank correlation over some periods of time for some fund groups. However, these results vary significantly across investment vehicles. The rank correlations were higher for mutual funds than for hedge funds – particularly for one five-year sample period. The results also differ through time. Thus the value of Sharpe ratios for performance ranking can be said to vary both through time and fund group.

Our main goal in constructing a utility-based measure is to solve the problem of manipulation highlighted in the first part of the paper. Our analysis of the differences in rank correlations across funds – particularly hedge funds – suggests that Sharpe ratio rankings can be quite sensitive to skewness. For mutual funds the skewness had both large positive and large negative effects on relative rank, depending upon time period. For hedge funds our evidence is at least consistent with the selling of out-of-the-money options over both periods of study.

## **5. Implications**

Our analysis has direct, practical implications for regulation, performance auditing and agency contracting. In this sense it relates to the growing literature on agency in money management (c.f. Chevalier and Ellison (1997), Carpenter (2000), and Goetzmann, Ingersoll and Ross (2001)). In settings in which the Sharpe ratio is used explicitly or implicitly for benchmarking, the use of options, or dynamic replication of derivative payoffs, may need to be constrained. Otherwise managers may take actions that do not coincide with their investors' interests. Further, it may pay those allocating assets to compare the distribution of high Sharpe ratio managers with those that can be obtained via an optimal manipulation strategy. In settings for which the use of options is unconstrained, asymmetric performance contracts similar to those used in the hedge fund industry appear to mitigate certain moral hazard problems raised by the use of Sharpe ratios.

It is likely that most portfolio modifications that remove some of the natural positive skewness and replace it with a balanced or negatively skewed portfolio will tend to increase the Sharpe ratio. This seems particularly relevant for market-neutral hedge funds, which, as a class, seem to have some of the least positively skewed distributions.

The analysis presented here applies to the use of Sharpe ratios in asset pricing. Low (1999) finds that large a class of U.S. equities has asymmetric exposure to the index. Glosten and Jagannathan (1994) liken this structural relationship to a derivative-based strategy. In effect, some assets in the U.S. market, primarily small cap stocks, behave as if they are short a put. Our analysis shows that in this case, the Sharpe and Information ratios are potentially biased measures of the attractiveness of an investment.

There may be corporate finance implications for these results as well. To the extent that a corporate manager is evaluated against an explicit benchmark, our strategy shows that there is an incentive to choose a capital structure that mimics the payoff of the MSRP. In the corporate setting, this would mean simultaneously issuing out-of-the-money call warrants and put warrants in a particular proportion. The former is common for certain types of firms. The latter is rare, but not unknown. Our paper provides at least one explanation for the existence of put warrants. In fact, even in settings where the corporate manager is evaluated not on stock returns but on the risk-scaled deviations of corporate earnings against a contemporaneous benchmark, our analysis suggests managers may smooth out large positive income realizations, while recognizing large negative hits.

The results presented here also have implications for the compensation structure typically found within the hedge fund industry. Most hedge fund's have a high-water mark incentive fee.<sup>31</sup> This fee is earned when the hedge fund's assets reach new levels. Since performance is measured after fees, this tends to truncate the upper tail of the distribution. Now it might seem obvious that any reduction of returns could only worsen the Sharpe ratio, but this is not the case. Removing the upper tail of the distribution (even with no compensation) can still improve the Sharpe ratio since it lowers the standard deviation more than proportionally to the mean. Thus, this particular compensation structure may lead hedge funds to score high relative to their benchmark even if they do nothing but add idiosyncratic risk to their portfolio returns!

In addition to the actual payments of hedge fund fees, the accounting for them can further distort the Sharpe ratio. The fee is usually paid annually, but returns are measured monthly or quarterly. A very high return in one period will be attenuated by the high-water mark fee accounting reserve for reporting purposes. If this good period is followed by another in which the fund loses money, then the fee is no longer earned and must be "rebated" to the fund from the reserve — increasing the reported return in the subsequent period. This means that very high returns tend to be truncated from the distribution and may also be used to smooth later reported returns. This understates the fund's risk and, therefore, overstates its Sharpe ratio. Thus, this accounting convention again makes it possible for a fund's Sharpe ratio to surpass its benchmark by having the manager add nothing but white noise to his returns.

The results also have implications for dynamic portfolio management. Brown, Harlow and Starks (1996) show that mutual fund managers increase variance after a poor showing in the first half of the year. While Busse (1999) disputes this evidence using daily data, our results in this paper suggest that this dynamic behavior is consistent with maximizing the Sharpe ratio. Brown, Goetzmann and Park (2000) find some evidence that

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<sup>31</sup> See Goetzmann, Ingersoll, and Ross (2003) for a description and analysis of high-water mark fees.

hedge fund managers increase volatility when they under-perform other funds, but not when they under-perform a fixed benchmark; thus, the existing empirical evidence is mixed.

The shape of the optimal Sharpe ratio leads to further conjectures. Expected returns being held constant, high Sharpe ratio strategies are, by definition, strategies that generate regular, modest profits punctuated by occasional crashes. Our evidence suggests that the “peso problem” may be ubiquitous in any investment management industry that rewards high Sharpe ratios.

## 6. Conclusion

This paper focuses on methods to manipulate portfolio returns to achieve high Sharpe ratios and related measures. It derives the optimal strategy under certain conditions and shows that the payoff structure resembles a portfolio that is short different fractions of out-of-the-money puts and calls, such that the fund distribution is left skewed. This result poses problems in the measurement and monitoring of investment funds and perhaps corporations in general because it distorts manager incentives.

Given the Sharpe ratio’s exposure to manipulation, we ask in this paper whether there exists, in effect, a “Sharper” Sharpe ratio that is not prone to manipulation using the strategies developed in this paper. We show that, if the uninformed representative agent in the economy wishes to hold the market portfolio, then a measure based upon the power utility function will penalize uninformed managers who try to manipulate it. Intuitively, if an agent (without private information) maximizes expected utility by holding the market portfolio then using this agent’s utility function to rank managers prevents the managers from gaming the measure. When it comes to judging managerial performance this has real potential benefits in practice as it discourages funds from making trades that do not add any economic value.

Our empirical analysis of this new measure compares historical rankings of mutual fund and hedge fund managers under the Sharpe ratio and under the utility-based measure. Since the utility-based measure is, by definition, optimal for a specific class of investors, our analysis is in effect a test of the performance of the Sharpe ratio. We find that it does a good job on average in ranking managers. However there are times when it performs poorly. For managers with skewed returns, we find that the Sharpe ratio ranking deviates dramatically from the optimal ranking. For hedge funds we find some evidence consistent with the hypothesis that some hedge fund managers may have enhanced Sharpe ratios through options-like strategies of the sort analyzed in this paper.

## Appendices

### A. Maximal-Sharpe-Ratio Portfolio in a Complete Market

Consider a portfolio with excess return  $x_i$  in state  $i$ . The probability of state  $i$  is  $p_i$ . The Sharpe ratio of this portfolio is

$$S = \frac{\sum p_i x_i}{\sqrt{\sum p_i x_i^2 - (\sum p_i x_i)^2}} \quad (\text{A1})$$

This is invariant to scaling so with no loss of generality we can fix the expected excess payoff at any positive value  $\sum p_i x_i = \bar{x} > 0$ . Then maximizing the Sharpe ratio of excess returns is equivalent to minimizing the mean squared payoff subject to an expected payoff of  $\bar{x}$  with a cost of zero.

Form the Lagrangian<sup>32</sup>

$$\mathcal{L} = \frac{1}{2} \sum p_i x_i^2 + \lambda (\bar{x} - \sum p_i x_i) + \gamma (\sum \hat{p}_i x_i) \quad (\text{A2})$$

The first-order conditions for a minimum are

$$0 = \frac{\partial \mathcal{L}}{\partial x_i} = p_i x_i^* - \lambda p_i + \gamma \hat{p}_i, \quad 0 = \frac{\partial \mathcal{L}}{\partial \lambda} = \bar{x} - \sum p_i x_i, \quad 0 = \frac{\partial \mathcal{L}}{\partial \gamma} = \sum \hat{p}_i x_i \quad (\text{A3})$$

The second-order condition for an interior minimum is also met.

Solving the first equation in (A3) gives the maximal-Sharpe-ratio return in state  $i$  as

$$x_i^* = \lambda - \gamma \hat{p}_i / p_i. \quad (\text{A4})$$

Multiply (A4) by  $p_i$  and  $\hat{p}_i$  and sum over states. Recognizing that  $\sum p_i = \sum \hat{p}_i = 1$ , gives

$$\begin{aligned} \bar{x} &= \sum p_i x_i^* = \lambda \sum p_i - \gamma \sum \hat{p}_i = \lambda - \gamma \\ 0 &= \sum \hat{p}_i x_i^* = \lambda \sum \hat{p}_i - \gamma \sum \hat{p}_i^2 / p_i = \lambda - \gamma \sum \hat{p}_i^2 / p_i \end{aligned} \quad (\text{A5})$$

These equations can be solved to determine the multipliers' values

$$\gamma = \frac{\bar{x}}{\sum \hat{p}_i^2 / p_i - 1} \quad \lambda = \bar{x} + \gamma \quad (\text{A6})$$

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<sup>32</sup> The zero-net-wealth budget constraint is expressed here using the risk-neutral probabilities in place of the state prices. Since the state price is  $e^{-rT} \hat{p}_i$ , a portfolio with a risk-neutral expected excess return of zero has a zero cost.

The variance of the “optimized” portfolio is

$$\begin{aligned} V &= \sum p(x_i^* - \bar{x})^2 = \sum p_i \gamma^2 (\hat{p}_i/p_i - 1)^2 \\ &= \gamma^2 \left[ \sum \hat{p}_i^2/p_i - 2 \sum \hat{p}_i + \sum p_i \right] = \gamma^2 \left[ \sum \hat{p}_i^2/p_i - 1 \right] = \gamma \bar{x}. \end{aligned} \quad (\text{A7})$$

Therefore, the square of the maximal Sharpe ratio<sup>33</sup> is

$$\mathcal{S}_*^2 = \frac{\bar{x}^2}{V} = \frac{\bar{x}}{\gamma} = \sum \hat{p}_i^2/p_i - 1. \quad (\text{A8})$$

This final sum can also be expressed as  $\sum p_i (\hat{p}_i/p_i)^2 = \mathbb{E}[(\hat{p}_i/p_i)^2]$ . Furthermore since  $\mathbb{E}[\hat{p}_i/p_i] = \sum \hat{p}_i = 1$ , we see that the maximal Sharpe ratio is equal to the variance across states of the likelihood ratio

$$\mathcal{S}_*^2 = \mathbb{E}[(\hat{p}_i/p_i)^2] - (\mathbb{E}[\hat{p}_i/p_i])^2 = \text{Var}[\hat{p}_i/p_i]. \quad (\text{A9})$$

Using (A6) and (A8) we can also express maximal-Sharpe-ratio in (A4) as

$$x_i^* = \bar{x} + \frac{\bar{x}}{\mathcal{S}_*^2} (1 - \hat{p}_i/p_i). \quad (\text{A10})$$

These are equations (4) and (3) in the text.

For a continuous state space indexed by  $b$ , similar analysis yields a maximal squared Sharpe ratio of

$$\mathcal{S}_*^2 = \int \hat{p}(b)/p(b) db - 1 = \mathbb{E}[\hat{p}^2(b)/p^2(b)] - 1 \quad (\text{A11})$$

for a portfolio with an excess return of

$$x^*(b) = \bar{x} + \frac{\bar{x}}{\mathcal{S}_*^2} [1 - \hat{p}(b)/p(b)]. \quad (\text{A12})$$

## B. Sharpe Ratios in a Lognormal Economy with Jumps

Suppose the evolution of wealth in the market portfolio is a lognormal diffusion with jumps

$$d(\ln b) = \mu' dt + \sigma d\omega + \tilde{\gamma} d\phi \quad (\text{B1})$$

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<sup>33</sup> Treynor and Black (1973) and some others call  $\mathcal{S}^2$  rather  $\mathcal{S}$  the Sharpe ratio. This is fine for an *ex ante* measure since a negative return can be converted to a positive return by shorting the portfolio. It has definite problems when applied *ex post*, however.

Let  $\Omega$  be the logarithm of the continuous portion of the change in wealth. (I.e.,  $d\Omega = \mu' dt + \sigma d\omega$ .) It has a normal distribution with mean  $\mu' \Delta t$  and variance  $\sigma^2 \Delta t$ .<sup>34</sup> The discontinuous portion of the return is directed by an independent Poisson process  $\phi$  with intensity (jump probability) of  $\lambda$  per unit time. The number of jumps,  $n$ , has a Poisson distribution. When a jump occurs the log of wealth changes by the random amount  $\tilde{\gamma}$  or wealth changes from  $W$  to  $\tilde{\Gamma} W$ , where  $\tilde{\Gamma} \equiv e^{\tilde{\gamma}}$ . We note for future reference the moment generating functions of the two distributions

$$\begin{aligned}\psi_{\Omega_T}(\theta) &\equiv \mathbb{E}[e^{\theta \tilde{\Omega}_T}] = \exp(\theta \mu' T + \frac{1}{2} \theta^2 \sigma^2 T) \\ \psi_{n_T}(\theta) &\equiv \mathbb{E}[e^{\theta \tilde{n}_T}] = \sum_{n=0}^{\infty} e^{\theta n} e^{-\lambda T} (\lambda T)^n / n! = \exp[(e^\theta - 1) \lambda T].\end{aligned}\tag{B2}$$

For expositional convenience we assume that the probability distribution of the jump amplitude is discrete; a jump of magnitude  $\Gamma_i$  occurs with probability  $\pi_i$ . We can therefore think of the process as distinct independent Poisson processes,  $n_i$ , for each jump amplitude with intensities of  $\lambda \pi_i$ .

Wealth at time  $T$  is  $W_T = W_0 \exp(\Omega_T + \sum_i \gamma_i n_{iT})$ . The expected rate of return on the market index,  $\mu$ , is

$$\mu \equiv \frac{1}{T} \ln \left[ \mathbb{E}[\exp(\tilde{\Omega}_T + \sum_i \tilde{n}_{iT} \gamma_i)] \right] = \frac{1}{T} \ln [\psi_{\Omega_T}(1) \prod_i \psi_{n_{iT}}(\gamma_i)] = \mu' + \frac{1}{2} \sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1).\tag{B3}$$

The variance of the market

$$\begin{aligned}\text{Var}[\exp(\tilde{\Omega}_T + \sum_i \tilde{n}_{iT} \gamma_i)] &= \mathbb{E}[\exp(2\tilde{\Omega}_T + 2\sum_i \tilde{n}_{iT} \gamma_i)] - e^{2\mu T} = \psi_{\Omega_T}(2) \prod_i \psi_{n_{iT}}(2\gamma_i) - e^{2\mu T} \\ &= \exp\left[2(\mu' + \sigma^2)T + \lambda T \sum_i \pi_i (\Gamma_i^2 - 1)\right] - \exp\left[2(\mu' + \frac{1}{2} \sigma^2)T + 2\lambda T \sum_i \pi_i (\Gamma_i - 1)\right] \\ &= e^{2\mu T} \left( \exp[\sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i - 1)^2] - 1 \right).\end{aligned}\tag{B4}$$

The square of the basis market's Sharpe ratio is

$$S_b^2 = \frac{(e^{\mu T} - e^{rT})^2}{e^{2\mu T} \left( \exp[\sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i - 1)^2] - 1 \right)} = \frac{(1 - e^{-(\mu-r)T})^2}{\exp[\sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i - 1)^2] - 1}.\tag{B5}$$

If the representative utility function is power,  $U(W_T) = e^{-\delta T} W_T^{1-\rho} / (1-\rho)$ , then marginal utility at time  $T$  is

$$U'(W_T) = e^{-\delta T} W_T^{-\rho} = e^{-\delta T} W_0^{-\rho} \exp\left[-\rho(\Omega_T + \sum_i n_{iT} \gamma_i)\right].\tag{B6}$$

<sup>34</sup> Note that  $\mu'$  is neither the expected rate of return on the market nor the expected log return. The expected rate of return is computed in (B3). The expected log return is  $\mu' + \lambda \sum_i \pi_i \gamma_i$ . Similarly the logarithmic variance is  $\frac{1}{T} \text{Var}[\tilde{\Omega}_T + \sum_i \tilde{n}_{iT} \gamma_i] = \sigma^2 + \lambda \sum_i \pi_i \gamma_i^2$  and not  $\sigma^2$ .

Expected marginal utility is

$$\begin{aligned}\mathbb{E}[U'(W_T)] &= e^{-\delta T} W_0^{-\rho} \mathbb{E}[-\rho(\tilde{\Omega}_T + \sum_i \tilde{n}_{iT} \gamma_i)] = e^{-\delta T} W_0^{-\rho} \Psi_{\Omega_T}(-\rho) \prod_i \Psi_{n_{iT}}(-\rho \gamma_i) \\ &= e^{-\delta T} W_0^{-\rho} \exp\left[-\rho \mu' T + \frac{1}{2} \rho^2 \sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i^{-\rho} - 1)\right].\end{aligned}\quad (\text{B7})$$

So the likelihood ratio is

$$\frac{\hat{p}(W_T)}{p(W_T)} = \frac{U'(W_T)}{\mathbb{E}[U'(W_T)]} = \exp\left[\rho(\mu - \frac{1}{2} \rho \sigma^2) T + \lambda T \sum_i \pi_i (1 - \Gamma_i^{-\rho}) - \rho(\Omega_T + \sum_i n_{iT} \gamma_i)\right]. \quad (\text{B8})$$

From (A11) the maximal Sharpe ratio is given by  $S_*^2 = \mathbb{E}[(\hat{p}/p)^2] - 1$  so

$$\begin{aligned}S_*^2 &= \exp\left[2\rho(\mu' - \frac{1}{2} \rho \sigma^2) T + 2\lambda T \sum_i \pi_i (1 - \Gamma_i^{-\rho})\right] \mathbb{E}\left[\exp\left(-2\rho(\tilde{\Omega}_T + \sum_i \tilde{n}_{iT} \gamma_i)\right)\right] - 1 \\ &= \exp\left[2\rho(\mu' - \frac{1}{2} \rho \sigma^2) T - 2\lambda T \sum_i \pi_i (\Gamma_i^{-\rho} - 1)\right] \Psi_{\Omega_T}(-2\rho) \prod_i \Psi_{n_{iT}}(-2\rho \gamma_i) - 1 \\ &= \exp\left(2\rho(\mu' - \frac{1}{2} \rho \sigma^2) T - 2\lambda T \sum_i \pi_i (\Gamma_i^{-\rho} - 1) - 2\rho \mu' T + 2\rho^2 \sigma^2 T + [\lambda T \sum_i \pi_i (\Gamma_i^{-2\rho} - 1)]\right) - 1 \\ &= \exp[\rho^2 \sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2] - 1.\end{aligned}\quad (\text{B9})$$

To compare these ratios we need the relation between the market risk premium,  $\mu - r$ , and the relative risk aversion,  $\rho$ . This can be determined from the representative investor's first order condition for optimally invested wealth  $\mathbb{E}[U'(W_T)(W_T - W_0 e^{rT})] = 0$

$$\begin{aligned}e^{rT} &= \frac{W_0^{1-\rho} \Psi_{\Omega_T} (1-\rho) \prod_i \Psi_{n_{iT}} ((1-\rho)\gamma_i)}{W_0^{1-\rho} \Psi_{\Omega_T} (-\rho) \prod_i \Psi_{n_{iT}} (-\rho \gamma_i)} = \frac{\exp\left[(1-\rho)\mu' T + \frac{1}{2}(1-\rho)^2 \sigma^2 T + \lambda T \sum_i \pi_i (\Gamma_i^{1-\rho} - 1)\right]}{\exp\left[(-\rho\mu' + \frac{1}{2} \rho^2 \sigma^2) T + \lambda T \sum_i \pi_i (\Gamma_i^{-\rho} - 1)\right]} \\ &= \exp\left[\mu' T + \frac{1}{2}(1-2\rho)\sigma^2 T + \lambda T \sum_i \pi_i \Gamma_i^{-\rho} (\Gamma_i - 1)\right].\end{aligned}\quad (\text{B10})$$

From (B10) and (B3), the market risk premium is therefore

$$\begin{aligned}\mu - r &= \mu' + \frac{1}{2} \sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1) - [\mu' + \frac{1}{2} (1-2\rho)\sigma^2 + \lambda \sum_i \pi_i \Gamma_i^{-\rho} (\Gamma_i - 1)] \\ &= \rho \sigma^2 + \lambda \sum_i \pi_i (1 - \Gamma_i) (\Gamma_i^{-\rho} - 1).\end{aligned}\quad (\text{B11})$$

Two special cases are of interest. In the no-jump case ( $\lambda = 0$ ), the two squared Sharpe ratios are

$$S_b^2 = \frac{[1 - e^{-(\mu-r)T}]^2}{e^{\sigma^2 T} - 1} \quad S_*^2 = \exp\left[\frac{(\mu-r)^2}{\sigma^2} T\right] - 1. \quad (\text{B12})$$

And for short periods the squared Sharpe ratios are approximately

$$\begin{aligned}
S_b^2 &= \frac{(\mu - r)^2}{\sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1)^2} T \left[ 1 - \left( \mu - r - \frac{1}{2} [\sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1)^2] \right) T \right] + o(T^2) \\
S_*^2 &= [\rho^2 \sigma^2 + \lambda \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2] T + \frac{1}{2} [\rho^2 \sigma^2 + \lambda \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2]^2 T^2 + o(T^2).
\end{aligned} \tag{B13}$$

Both are zero in the limit  $T \rightarrow 0$ ; however, the maximal Sharpe ratio is larger to  $o(T)$  since<sup>35</sup>

$$\begin{aligned}
\frac{S_b^2}{T} &= \frac{\rho^2 \sigma^4}{\sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1)^2} + \frac{\lambda^2 \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2 (\Gamma_i - 1)^2}{\sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1)^2} - \frac{2\rho \sigma^2 \lambda \sum_i \pi_i (\Gamma_i^{-\rho} - 1) (\Gamma_i - 1)}{\sigma^2 + \lambda \sum_i \pi_i (\Gamma_i - 1)^2} + O(T) \\
&< \rho^2 \sigma^2 + \lambda \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2 - 0 + O(T) = \frac{S_*^2}{T} + O(T).
\end{aligned} \tag{B14}$$

### C. Maximal-Sharpe-Ratio Portfolio with Puts and Calls

Let  $z$  be normally distributed  $\mathcal{N}(\bar{z}, v^2)$ . Let  $Z \equiv e^z$ , then  $Z$  is lognormally distributed with

$$\begin{aligned}
\mathbb{E}[Z] &= \mathbb{E}[e^z] = \exp\left(\bar{z} + \frac{1}{2} v^2\right) \\
\text{Prob}\{Z > K\} &= \Phi\left(\frac{\bar{z} - \ln K}{v}\right) \\
\int_K^\infty Z dF(Z) &= \exp\left(\bar{z} + \frac{1}{2} v^2\right) \Phi\left(\frac{\bar{z} - \ln K + v^2}{v}\right)
\end{aligned} \tag{C1}$$

where  $\Phi(\cdot)$  is the standard cumulative normal function. All this is standard for the Black-Scholes model where  $Z$  represents the stock price at maturity  $S_T$  and so  $\bar{z} = \ln S + (\mu - \frac{1}{2} \sigma^2) T$  and  $v^2 = \sigma^2 T$ . The two additional results we need are the upper and lower noncentral truncated moments of  $Z$ .

$$\begin{aligned}
\int_K^\infty Z^\zeta dF(Z) &= \exp\left(\zeta \bar{z} + \frac{1}{2} \zeta^2 v^2\right) \Phi\left(\frac{\bar{z} - \ln K + \zeta v^2}{v}\right) \\
\int_0^K Z^\zeta dF(Z) &= \exp\left(\zeta \bar{z} + \frac{1}{2} \zeta^2 v^2\right) \Phi\left(-\frac{\bar{z} - \ln K + \zeta v^2}{v}\right).
\end{aligned} \tag{C2}$$

Note that the three expressions in equation (C1) are all special cases of the first line in (C2). The first line is  $\zeta = K = 0$ . The second and third lines are  $\zeta = 0$  and  $\zeta = 1$ .

The proof is straightforward. Let  $W \equiv Z^\zeta$ , then  $w \equiv \ln W$  is normally distributed  $\mathcal{N}(\zeta \bar{z}, \zeta^2 v^2)$ . And applying the third line in (C1) we get

<sup>35</sup> The first fraction is smaller than  $\rho^2 \sigma^2$  since  $\lambda$ ,  $\pi_i$  and  $(\Gamma_i - 1)^2$  are positive. The third fraction is positive because each term is positive;  $\Gamma_i^{-\rho}$  is less (greater) than one implies  $\Gamma_i$  is greater (less) than one. For the second fraction note that  $\sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2 (\Gamma_i - 1)^2 = \mathbb{E}[(\Gamma - 1)^2] \mathbb{E}[(\Gamma^{-\rho} - 1)^2] + \text{Cov}[(\Gamma^{-\rho} - 1)^2 (\Gamma - 1)^2]$ . The final covariance is negative since  $\Gamma$  and  $\Gamma^{-\rho}$  are inversely related; therefore  $\sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2 (\Gamma_i - 1)^2 < \sum_i \pi_i (\Gamma_i^{-\rho} - 1)^2 \sum_i \pi_i (\Gamma_i - 1)^2$ .

$$\int_K^\infty Z^\zeta dF(Z) = \int_{K^\zeta}^\infty W dF_w(W) = \exp\left(\zeta\bar{z} + \frac{1}{2}\zeta^2 v^2\right) \Phi\left(\frac{\zeta\bar{z} - \zeta \ln K + \zeta^2 v^2}{\zeta v}\right) \quad (\text{C3})$$

which reduces to the first line in (C2). The second line follows by complementarity.

#### D. The Sharpe Ratio for a Simple Model of Information<sup>36</sup>

Suppose a money manager has superior information. As a simple case suppose he knows that the price of some asset deviates from the fair value by an amount  $\varepsilon$  and that this error tends to disappear over time. Let  $\alpha(\Delta t)$  and  $\Gamma(\Delta t)$  be the expected change and variance in the error term over the interval  $\Delta t$ . Assume otherwise that the CAPM holds and that this asset has a residual variation of  $s^2$  per unit time (in addition to the error risk). For simplicity assume that the error risk is uncorrelated with both the market and the stock's "normal" idiosyncratic risk.

The informed investor will hold the market portfolio and a position in the mispriced asset. For simplicity, assume that the asset has a zero beta with the market (if not construct a new artificial asset holding a short position (or long position if the asset's beta is negative) in the market to offset the systematic risk. The stock's excess expected rate of return and variance are then  $\alpha(\Delta t)$  and  $s^2\Delta t + \Gamma(\Delta t)$ .

The Sharpe-ratio-maximizing portfolio over interval  $\Delta t$  holds the market and the asset in proportions:

$$w_m = \frac{\mu_m}{\sigma_m^2} \quad w_i = \frac{\alpha(\Delta t)}{s^2\Delta t + \Gamma(\Delta t)} \quad (\text{D4})$$

where  $\mu_m$  and  $\sigma_m^2$  are the expected excess return on the market and its variance. The squared Sharpe ratio for the informed investor is

$$\begin{aligned} S_{\text{inf}}^2 &= \frac{[w_m \mu_m \Delta t + w_i \alpha(\Delta t)]^2}{w_m^2 \sigma_m^2 \Delta t + w_i^2 [s^2 \Delta t + \Gamma(\Delta t)]} \\ &= \left[ \frac{\mu_m^2}{\sigma_m^2} \Delta t + \frac{\alpha^2(\Delta t)}{s^2 \Delta t + \Gamma(\Delta t)} \right]^2 \left[ \frac{\mu_m^2}{\sigma_m^2} \Delta t + \frac{\alpha^2(\Delta t)}{[s^2 \Delta t + \Gamma(\Delta t)]^2} [s^2 \Delta t + \Gamma(\Delta t)] \right]^{-1} \\ &= \frac{\mu_m^2}{\sigma_m^2} \Delta t + \frac{\alpha^2(\Delta t)}{s^2 \Delta t + \Gamma(\Delta t)}. \end{aligned} \quad (\text{D5})$$

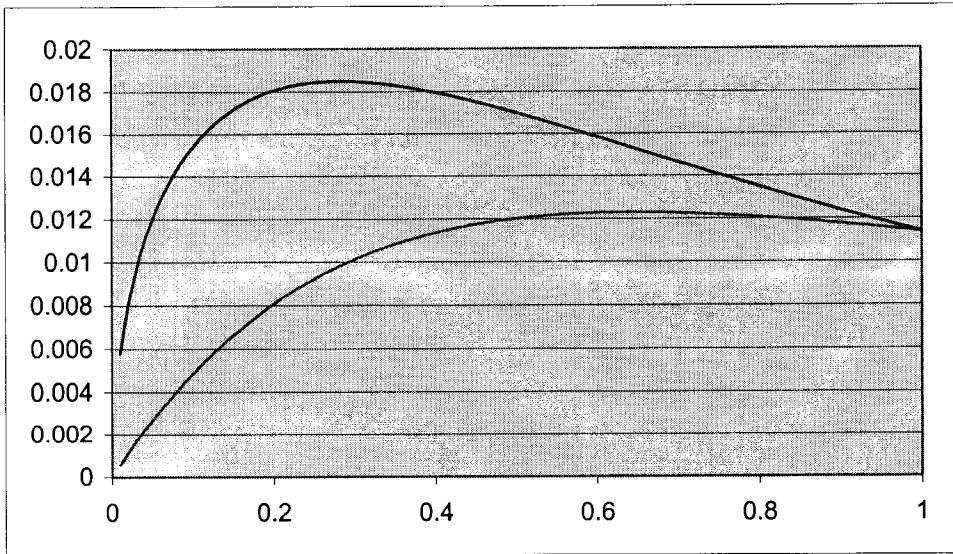
A simple assumption for the error is linearly mean reverting (i.e.,  $d\varepsilon = -\kappa\varepsilon dt + \gamma d\omega$ ). In this case,  $\alpha(\Delta t) = \varepsilon(1 - e^{-\kappa\Delta t})$  and  $\Gamma(\Delta t) = \gamma^2(1 - e^{-2\kappa\Delta t})/2\kappa$ . The squared Sharpe ratio over a period of  $\Delta t$  conditional on the known error  $\varepsilon$  is

<sup>36</sup> This problem is identical to that analyzed in Ferson and Siegel (2001). We have specialized it to examine the effects of different horizons.

$$S_{\text{inf}}^2 = \frac{\mu_m^2}{\sigma_m^2} \Delta t + \frac{\varepsilon^2 (1 - e^{-\kappa \Delta t})^2}{s^2 \Delta t + (\gamma^2 / 2\kappa) (1 - e^{-2\kappa \Delta t})}. \quad (\text{D6})$$

On average the asset will be mispriced by  $\mathbb{E}[\varepsilon^2] = \gamma^2 / 2\kappa$ .

The chart below shows the difference between the informed and basis Sharpe ratios  $S_{\text{inf}} - S_m$  (upper curve) and the “normalized” difference  $(S_{\text{inf}} - S_m) / \sqrt{\Delta t}$  for different observation periods and for parameter values  $\kappa = 2$ ,  $s = 20\%$ ,  $\gamma = 5\%$ ,  $\mu_m = 10\%$ ,  $\sigma_m = 20\%$ . Clearly using very short observation periods is not the best way to detect superior performance in this model.



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**Table I: Annualized Sharpe Ratios in a Mixed Jump Lognormal Diffusion Economy**

**Jumps with  $\Gamma = 0.95, \lambda = 0.5$**

	$T = 1$ year				$T = 1$ month			
	log vol.	$\mu - r$			log vol.	$\mu - r$		
		5%	10%	15%		5%	10%	15%
Maximal Sharpe $S_*$ eq. (8)	15%	0.350	0.784	1.468	15%	0.341	0.699	1.098
	20%	0.256	0.541	0.891	20%	0.253	0.509	0.774
	25%	0.203	0.419	0.665	25%	0.201	0.404	0.609
Basis Sharpe $S_b$ eq. (10)	15%	0.327	0.639	0.935	15%	0.337	0.672	1.005
	20%	0.243	0.474	0.694	20%	0.251	0.501	0.750
	25%	0.193	0.376	0.551	25%	0.200	0.400	0.598
apparent alpha in basis pts. eq. (12) – eq. (9)	15%	35.7	243.0	969.3	15%	6.4	41.6	139.5
	20%	27.9	148.0	469.1	20%	3.2	16.3	48.3
	25%	27.1	120.8	339.4	25%	2.5	10.8	28.6

**No Jumps ( $\lambda = 0$ )**

	$T = 1$ year				$T = 1$ month			
	log vol.	$\mu - r$			log vol.	$\mu - r$		
		5%	10%	15%		5%	10%	15%
Maximal Sharpe $S_*$ eq. (8)	15%	0.343	0.748	1.311	15%	0.096	0.194	0.295
	20%	0.254	0.533	0.869	20%	0.072	0.145	0.219
	25%	0.202	0.417	0.658	25%	0.058	0.116	0.175
Basis Sharpe $S_b$ eq. (10)	15%	0.323	0.631	0.923	15%	0.096	0.192	0.287
	20%	0.241	0.471	0.690	20%	0.072	0.144	0.215
	25%	0.192	0.375	0.548	25%	0.058	0.115	0.172
apparent alpha in basis pts. eq. (12) – eq. (9)	15%	31.0	197.4	703.2	15%	2.4	14.1	42.4
	20%	26.7	139.1	430.3	20%	2.1	10.3	28.7
	25%	26.7	118.1	329.3	25%	2.1	8.9	22.9

The annualized Sharpe ratio is  $S/\sqrt{T}$  where the Sharpe ratio is computed from equation (8) or (10). The apparent alpha is computed as the difference between the apparent risk premium in equation (12) and the actual risk premium in equation (9). The logarithmic volatility is  $\sigma^2 + \lambda \sum_i \pi_i (\ln \Gamma_i)^2$

**Table II: Skewness and Kurtosis of Maximal Sharpe Ratio Portfolio**

		<u><math>T = 1</math> year</u>						<u><math>T = 1</math> month</u>			
$\sigma$	Skewness eq. (15)		Kurtosis eq. (15)		$\sigma$	Skewness eq. (15)		Kurtosis eq. (15)			
	<u>Basis</u>	<u>MSRP</u>	<u>Basis</u>	<u>MSRP</u>		<u>Basis</u>	<u>MSRP</u>	<u>Basis</u>	<u>MSRP</u>		
15%	0.456	-2.663	3.372	17.801	15%	0.130	-0.590	3.030	3.625		
20%	0.614	-1.750	3.678	8.898	20%	0.174	-0.438	3.054	3.344		
25%	0.778	-1.322	4.096	6.260	25%	0.217	-0.349	3.084	3.217		

**Table III: Rank Correlations Across Measures**

This table shows the rank correlations between the Sharpe ratio and the manipulation-free measure ( $\rho = 0, 1, 2, 6$ ) for mutual funds in the CRSP monthly mutual fund database and hedge funds in the TASS database and the market portfolio's percentile performance for each measure. E.g., the rank correlation across all funds between the manipulation-free ( $\rho = 2$ ) measure and Sharpe ratio was 0.986. The market portfolio beat 82.6% of all mutual funds based on the Sharpe ratio and 83.2% of all mutual funds based on the manipulation-free measure. For the mutual funds, the rank correlations are also given relative to the funds in their own objective category. The columns  $\rho = 0$  and  $\rho = 1$  rank funds on arithmetic average return alone and on total return per dollar over the five-year period alone ignoring risk.

**Rank Correlation between Manipulation-Free Measure and Sharpe Ratio  
and  
The Market's Percentile Performance within Class of Funds**

<b>CRSP Mutual Funds Oct 1993 – Sept 1998</b>						
Category	Sharpe	$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 6$	N
All Mutual Funds	82.6%	0.951 82.4%	0.973 83.3%	0.986 83.2%	0.966 82.2%	1008
Aggressive Growth	99.5%	0.946 95.3%	0.985 97.4%	0.992 98.9%	0.896 99.5%	190
Global Equity	94.5%	0.960 94.5%	0.981 94.5%	0.993 94.5%	0.963 94.5%	55
Growth and Income	52.2%	0.942 63.6%	0.958 62.0%	0.972 59.8%	0.995 49.5%	184
International Equities	96.8%	0.996 96.1%	0.991 96.1%	0.983 95.5%	0.961 96.1%	155
Income	64.3%	0.818 96.4%	0.860 96.4%	0.918 92.9%	0.993 60.7%	28
Long-Term Growth	78.8%	0.910 75.9%	0.948 77.5%	0.974 77.5%	0.985 79.4%	311
Precious Metals	94.4%	0.967 94.4%	0.818 94.4%	0.695 94.4%	0.304 94.4%	18
Sector Funds	86.0%	0.955 64.9%	0.971 70.2%	0.982 73.7%	0.879 89.5%	57
Utility Funds	94.4%	0.950 83.3%	0.953 83.3%	0.965 83.3%	0.971 88.9%	18
<b>TASS Hedge Funds Oct 1992 – Sept 1997</b>						
	Sharpe	$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 6$	N
All Hedge Funds	77.6%	0.595 74.5%	0.680 76.6%	0.748 78.3%	0.895 84.7%	411

**Table III: Rank Correlations Across Measures (part 2)**

**Rank Correlation between Manipulation-Free Measure and Sharpe Ratio  
and  
The Market's Percentile Performance within Class of Funds**

<b>CRSP Mutual Funds Oct 1998 – Sept 2003</b>						
<b>Category</b>	<b>Sharpe</b>	<b><math>\rho = 0</math></b>	<b><math>\rho = 1</math></b>	<b><math>\rho = 2</math></b>	<b><math>\rho = 6</math></b>	<b>N</b>
All Mutual Funds	42.7%	0.981 42.5%	0.962 45.8%	0.886 49.5%	0.552 55.7%	3248
Aggressive Growth	15.0%	0.919 14.7%	0.982 21.5%	0.964 31.1%	0.787 53.6%	673
Global Equity	53.8%	0.986 53.8%	0.918 54.4%	0.865 57.4%	0.561 60.4%	169
Growth and Income	65.0%	0.995 65.2%	0.970 64.3%	0.923 64.3%	0.762 52.2%	515
International Equities	48.6%	0.993 48.6%	0.986 47.8%	0.966 48.9%	0.664 58.6%	556
Income	61.4%	0.998 62.5%	0.992 54.5%	0.973 51.1%	0.796 22.7%	88
Long-Term Growth	53.1%	0.992 52.5%	0.967 56.8%	0.894 59.8%	0.577 65.0%	973
Precious Metals	0.0%	0.965 0.0%	0.987 4.8%	0.990 9.5%	0.708 57.1%	21
Sector Funds	11.5%	0.797 11.9%	0.919 23.4%	0.901 30.5%	0.745 39.8%	269
Utility Funds	90.7%	0.923 90.7%	0.831 90.7%	0.736 90.7%	0.489 74.4%	43
<b>TASS Hedge Funds Oct 1997 – Sept 2002</b>						
	<b>Sharpe</b>	<b><math>\rho = 0</math></b>	<b><math>\rho = 1</math></b>	<b><math>\rho = 2</math></b>	<b><math>\rho = 6</math></b>	<b>N</b>
All Hedge Funds		0.765	0.848	0.881	0.856	799
	9.3%	8.8%	10.5%	13.5%	18.4%	

**Table IV: Effect of Skewness on Relative Performance**

Regression results for the performance percentile difference between the Sharpe ratio and the manipulation-free measure as the dependent variable and normalized return skewness as the independent variable. The normalized return skewness is the third moment of the return distribution divided by the cube of the standard deviation. The slope coefficient  $\beta$  is reported with t-statistics in parenthesis.

$$(\text{Sharpe \%tile})_i - (\text{Manipulation-Free \%tile})_i = \alpha + \beta \times \text{Skewness}_i + \varepsilon_i$$

	CRSP Mutual Funds									
	Oct 1993 – Sep 1998					Oct 1998 – Sep 2003				
	$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 6$	N	$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 6$	N
All Mutual Funds	-1.90 (-3.37)	-1.14 (-2.71)	-0.63 (-2.08)	-1.07 (-2.26)	1008	-3.91 (-13.59)	0.37 (0.88)	5.70 (7.95)	21.33 (15.35)	3248
Aggressive Growth	-10.58 (-6.24)	-3.08 (-3.24)	1.69 (2.41)	10.33 (4.18)	190	-14.22 (-13.78)	-2.88 (-5.35)	4.65 (6.13)	15.64 (8.72)	673
Global Equity	-2.63 (-1.02)	-2.44 (-1.36)	-2.11 (-1.99)	0.51 (0.20)	55	-1.35 (-1.25)	-0.52 (-0.20)	2.55 (0.76)	22.05 (3.79)	169
Growth and Income	8.71 (2.68)	7.41 (2.66)	5.18 (2.27)	-6.18 (-7.48)	184	-2.36 (-2.89)	5.39 (2.57)	10.43 (3.14)	6.72 (1.14)	515
International Equities	-1.22 (-2.48)	-2.66 (-3.78)	-3.58 (-3.62)	-5.99 (-4.02)	155	-1.49 (-4.59)	1.95 (4.20)	5.37 (7.80)	19.55 (9.16)	556
Income	23.31 (2.79)	19.16 (2.57)	17.73 (3.29)	-4.97 (-3.07)	28	-0.25 (-0.20)	6.49 (2.64)	12.71 (2.83)	34.41 (2.76)	88
Long-Term Growth	-11.06 (-5.22)	-7.88 (-4.87)	-5.27 (-4.60)	0.31 (0.34)	311	-3.73 (-8.10)	-3.23 (-3.42)	-0.63 (-0.37)	10.02 (2.95)	973
Precious Metals	-0.36 (-0.14)	2.40 (0.39)	3.38 (0.43)	7.05 (0.60)	18	-14.11 (-1.95)	-3.69 (-0.78)	9.29 (2.47)	78.31 (5.51)	21
Sector Funds	-6.94 (-3.37)	-6.25 (-3.92)	-5.41 (-4.45)	-5.32 (-1.47)	57	1.96 (0.59)	-8.14 (-4.00)	-5.31 (-2.32)	5.64 (1.52)	269
Utility Funds	1.96 (0.44)	2.48 (0.57)	-0.95 (-0.25)	-9.39 (-3.79)	18	-24.22 (-3.52)	-27.96 (-2.60)	-30.85 (-2.25)	-24.27 (-1.22)	43

	TASS Hedge Funds									
	Oct 1992 – Sep 1997					Oct 1997 – Sep 2002				
	$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 6$	N	$\rho = 0$	$\rho = 1$	$\rho = 2$	$\rho = 6$	N
All Hedge Funds	-3.72 (-2.19)	-3.45 (-2.28)	-3.16 (-2.35)	-0.21 (-0.23)	411	-3.44 (-6.33)	-2.82 (-6.47)	-1.91 (-4.91)	1.02 (-2.34)	799

## Comparison of Maximal Sharpe and Optimal Portfolios

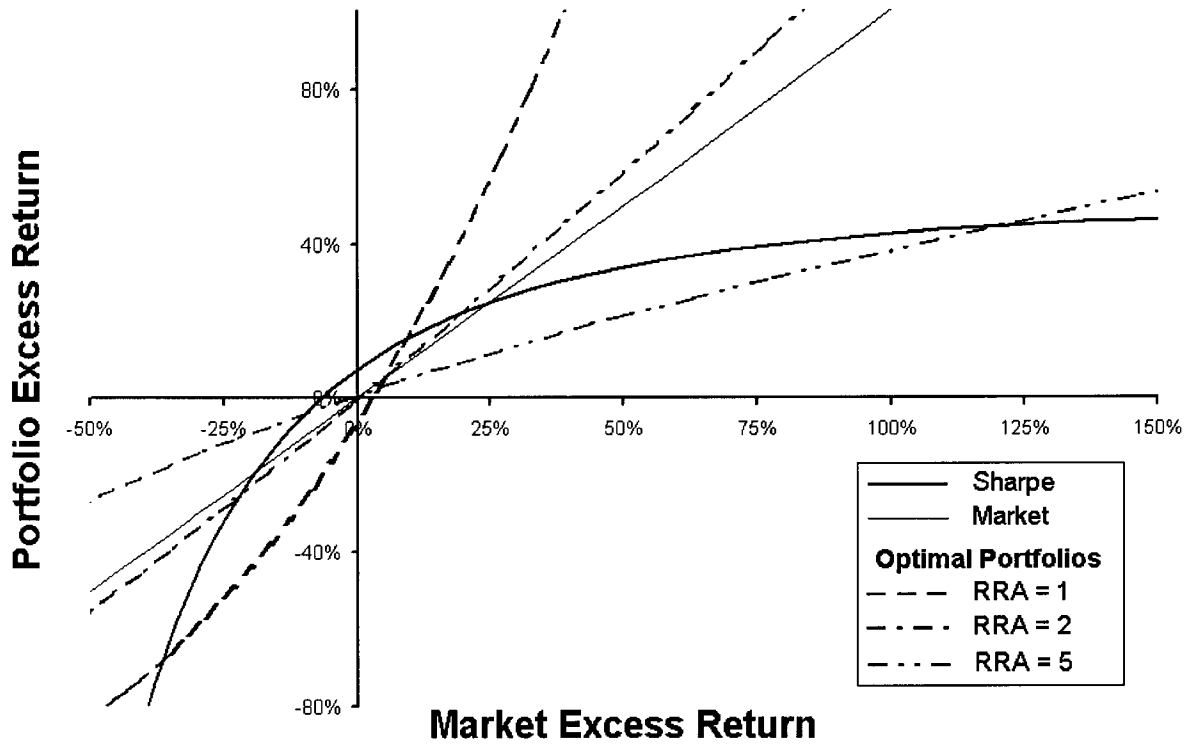
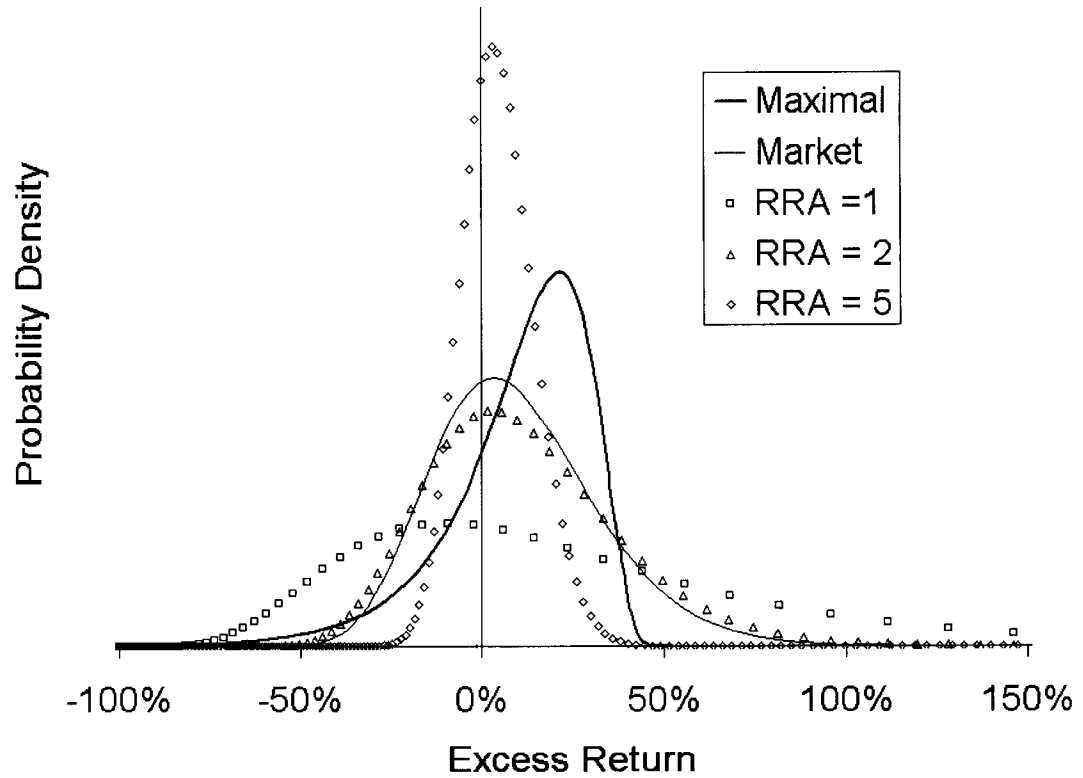
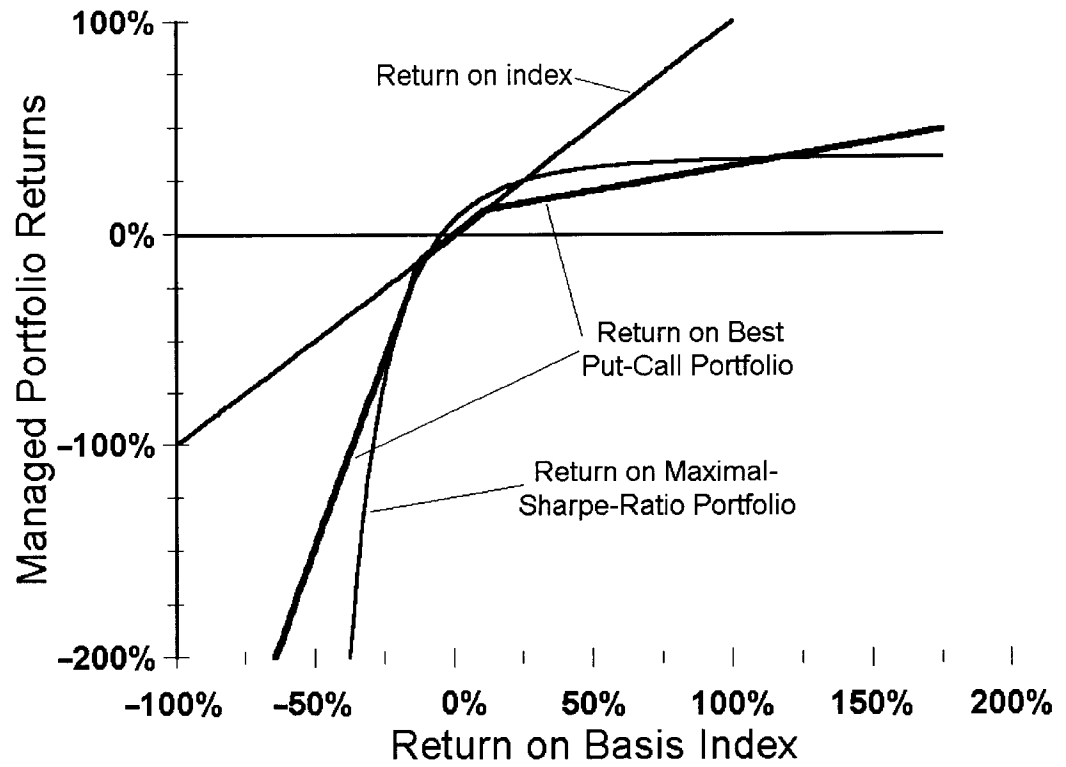


Figure 1. Comparison of Maximal-Sharpe-Ratio Portfolio with Market Basis and Optimal Portfolios

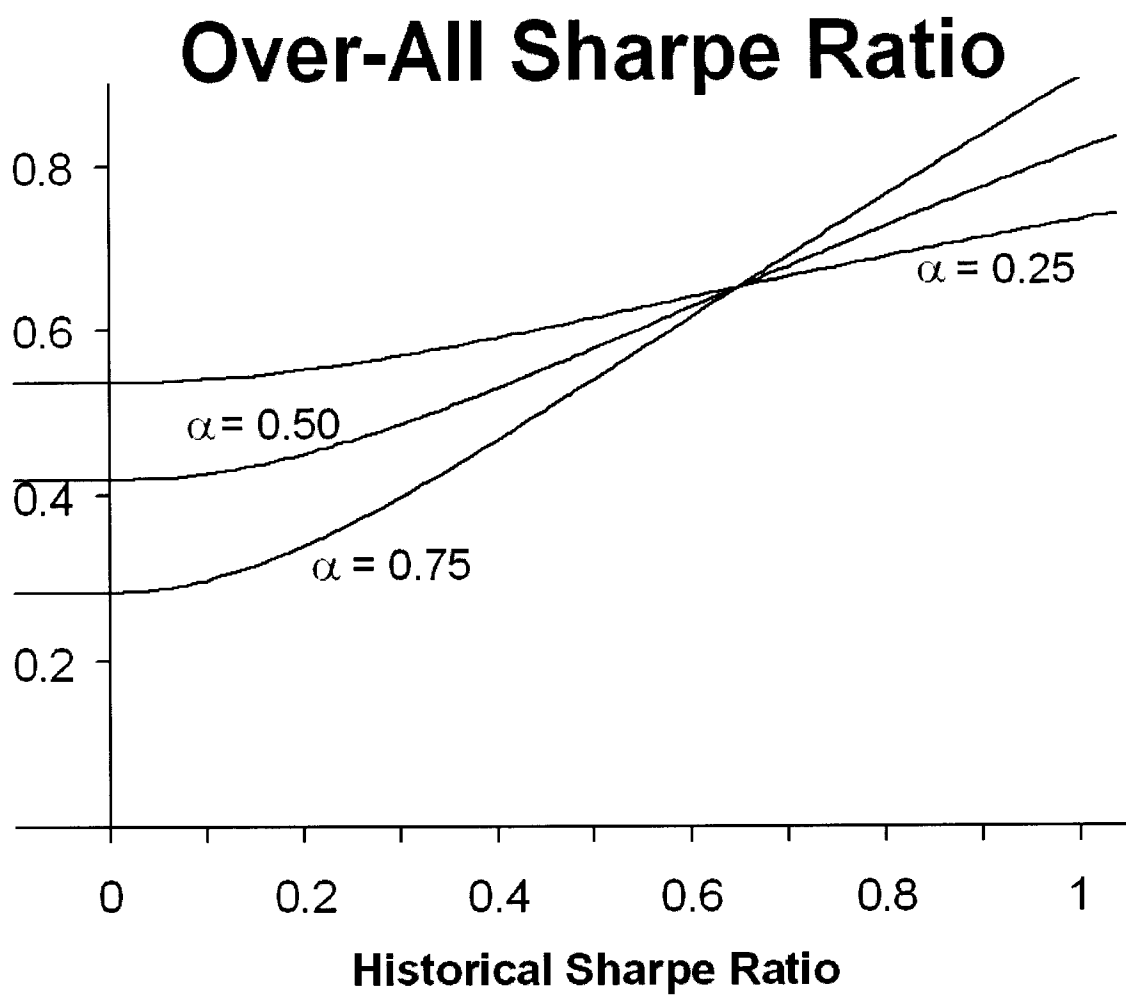
## Distributions of Portfolio Returns



**Figure 2: Probability Distributions of Maximal-Sharpe-Ratio Portfolio, Basis, and Optimal Portfolios**



**Figure 3: Maximal-Sharpe-Ratio Portfolio Return Holding Stock, Put, and Call.**



**Figure 4: Dynamic Optimization of the Sharpe Ratio**